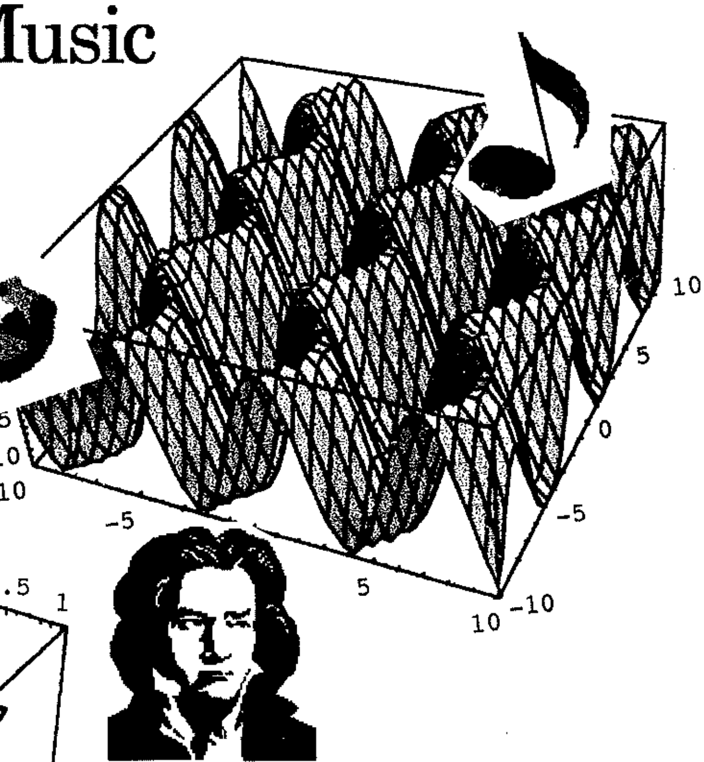
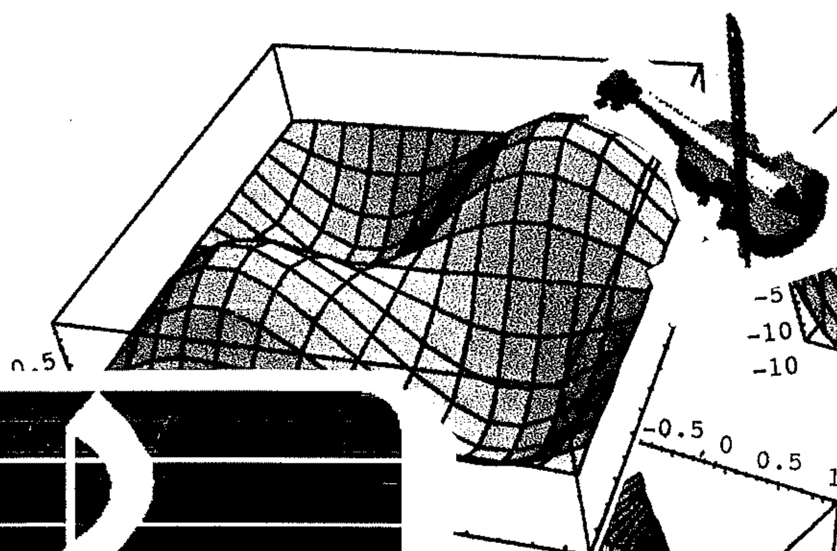
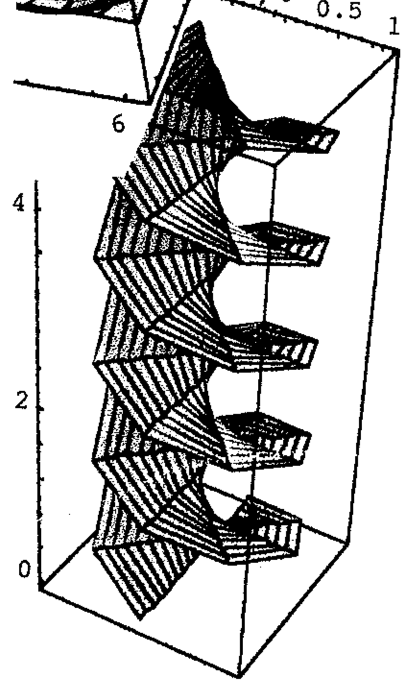
Scales, Tuning, Philosophy, Number theory and Frequency:
Math in Music







Sara Smollett
May 26, 1995
Antilles School

Mr. Button

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Thanks

I'd like to thank all of the people on the Internet who responded to my posts personally, but my report is long enough as it is. A few people I'd like to especially thank: Laura Helen (who is still being hassled about her "over sensitiveness"), Douglas Zare (for being one of the few people to take my first post seriously), Dave Buck, Megan Staples (who was willing to talk extensively on the subject), Dave Rusin (who led me to his web site), Bill Atkerson, and everyone (all thirteen of them) who suggested *Gödel, Escher, and Bach*, which although it added little to this report was a fun read. Special thanks to Dana Albrecht for e-mailing me a compiling of previous posts on the subject and Edelbar for mailing me an article. Thanks to my parents for their co-operation and patience even though this report didn't come up at the best of times. Thanks to Mr. Lakins for teaching his awesome music history class and making his personal library available. Also thanks to Norman Messa for giving me feedback, keeping in touch with me since my last report, and taking an interest in whatever it is that I am doing. Thanks to the caffeine that helped me stay up late at night. Thanks to those who allowed me use of their photocopiers in desperate time of need. Thanks to Linda and Phebe for help with the layout. Thanks to Pythagoras, Bach, Mozart, Schoenberg, and many others for making music what it is. Thanks to Her Majesty the Yellow Pig and Kelly (who is going to tell me sometime soon that I can return to Hampshire, right?).

Although the topic of this report started out as trigonometry in music, it quickly expanded to cover philosophy of music, the relationships of music and math, math at its play in sound, and some music history. This is (believe it or not) a revised and shortened edition. I have tried not to take sides during my discussions of music and math, but probably not very successfully. Enjoy.

Scales, Tuning, Philosophy, Number theory and Frequency: Math in Music

Introduction

Almost two months ago, I started thinking about things that trigonometry is related to, and there was of course, a seemingly endless list. On one extreme was trig in nature (that is, how trigonometry is found naturally in life), and on the other end of the spectrum was sine, cosine, and tangent in applied sciences and manmade objects. I chose to steer somewhere toward what is, I hope, the middle – music. Music is both naturally occurring and manmade; it is both something extremely formulated and academic in some aspects and elegant with sheer beauty in the others; music is an art.

As a general trend, there are a large number of people who excel in music who also do quite well in other types of math (Pythagoras and his followers considered music to be a type of math, but for the purpose of not making this paper any more confusing then it will get anyway, I'll refer to it as an art, a category under which it and math can both fall). There have also been studies of listening to classical music enhancing and stimulating intellectual and creative activities. The study of the relationships between math, music, logic, psychology, and beauty seems to be a deep-rooted one.

Music and math both have many things in common. I already mentioned that music was an art, and I am now going to classify math as one as well, the art of logical thought and elegant proofs. Also, music can be identified as something mainly to be appreciated for beauty or as something to be analyzed in form and structure. Math is very similar in the divisions that exist between pure mathematicians who see their work as art and applied mathematicians who search to find applications of math.

Because this can be a very controversial issue (at least to those who spend time flaming others in sci.math), I was able to find much interesting information on math (trig) and acoustics (music), but a majority of it is not pertinent to anything (except those flame groups). Because the opinions of people who chose to reply varied so much and touch on this subject, I have chosen to include a handful of replies as an appendix to this report.

Also, I have found many books that discuss the psychological aspects of music in great detail and will include some of the thoughts of great mathematicians, musicians and composers, and philosophers on how music affects the mind. I hope to touch on both the applications of trigonometric functions to music and on the issue of how to interpret music and math in this report.

Parts of this report may digress from math and music and go off into psychological affects and the divisions between schools of music, but I figure that I have not gone too far off topic, because after all, tangents are trigonometric.

Relationships between music and math

First, I'm going to explain some of the more obvious relationships between music and math, regardless of how basic or trivial some of them may seem.

Notes are given not only a specific pitch, but also a time length. In 4/4 time (Time signatures are of the form: number of beats to a measure over the type of note that gets one beat. In 4/4 time there are four beats to a measure and a quarter note gets one beat, so a whole note gets four full counts.) Because there can only be four beats in a measure, there can only be one whole note in a measure. Half notes are, as their name suggests, given half the beat of a whole note, or in this case two counts. Quarter notes are given one beat in 4/4 time. There are also eighth notes, sixteenth notes, and more rarely thirty-second notes and so on. This exponential sequence is powers of one-half.

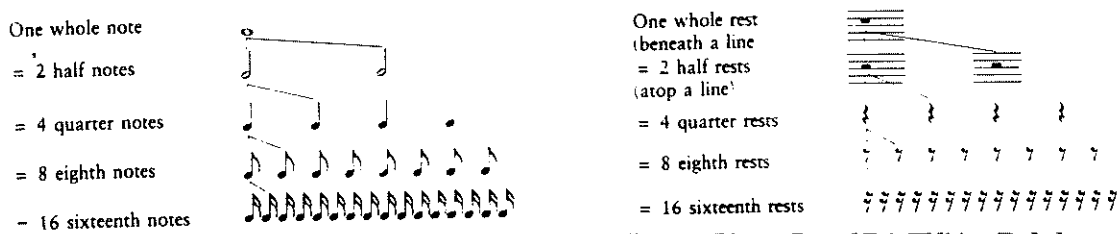

Fig. 1. Notes and Rests -- powers of two. Joseph Kerman, <u>Listen: Second Brief Edition</u> (Berkeley, California: Worth Publishers, 1992) 24.

There are three other things that need to be discussed here. One, ties can be used to connect two or more notes that may be in different measures so it is possible for the duration of a note to be more than a whole note. Two, in music theory there is the concept of dotted notes. Dotted notes are held for one and a half times their usual length. So the commonly used dotted half note in 4/4 time is held for three beats. Three, sometimes there is a slight variation in the length of a note -- staccato notes (marked with a dot above them) are to be played quickly and detached, slurs connect notes to be played smoothly which tend to blend into one another, and triplets (three notes grouped together and marked), are played one-third faster than usual.

Time signatures are also very mathematical; composers have experimented with different ratios to develop ones which sound most euphonic to the ear and those which will evoke certain feelings. 4/4 time, 2/2 time, 6/8 time, and 3/4 time (waltz time) are commonly used. Examples of less common time signatures or combinations of time signatures (as this may modulate within a piece) include Mussorgsky's *Pictures at an Exhibition*, "Promenade" which alternates between 5/4 and 6/4.[1]

Today there are two modes commonly used, the major (C) and the minor (A) modes. Originally, there was one for each of the notes, but some of them proved to be inharmonious and so they were, for the most part, discarded.

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[1] Joseph Kerman, <u>Listen: Second Brief Edition</u> (Berkeley, California: Worth Publishers, 1992) 306.

Here are two interesting things to note in music theory about scales, modes, and keys. If you start with key of C, there are no sharps or flats; five notes away is G, and in the key of G there is one sharp; the pattern of adding successive sharps and flats continues as you go another five white notes up and is known as the circle of fifths. Related to this is the fact that to find the notes that are in a scale in the major mode, you begin on the note of the scale and use the pattern: whole step - whole step - half step - whole step - whole step - whole step - half step (in the minor scale the pattern is whole step - half step - whole step - whole step - half step - whole step - whole step).

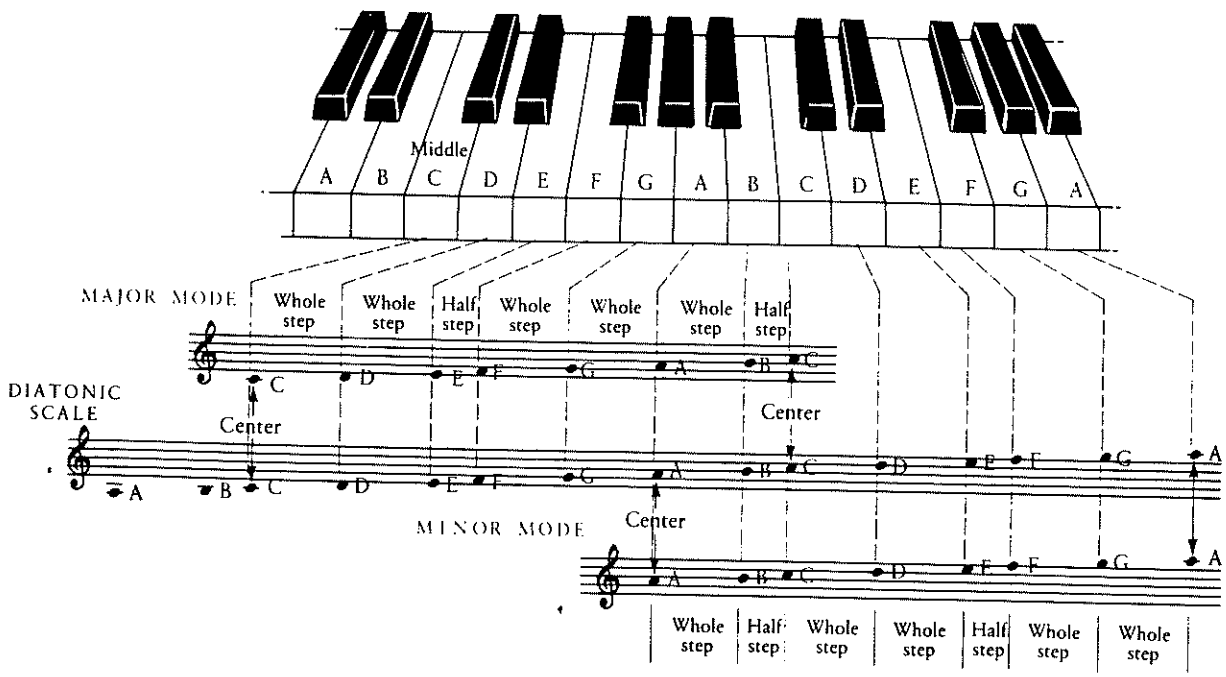


Fig. 2. Major and Minor Modes. Joseph Kerman, Listen: Second Brief Edition (Berkeley, California: Worth Publishers, 1992) 38.

As early as 500 B.C., music was considered one of the four subjects under the broader heading of math. Pythagoras, known in math primarily for the theorem which bears his name, was instrumental (no pun intended) in tuning and the development of scales.

In Medieval music, modes were used instead of scales. Modes are centered on a home pitch, and they modulate, or change. The mode is the internal difference with intervals.[2]

[2] Joseph Kerman, 38.

A few terms to be aware of

•Pitch is the quality of the highness or lowness of sound.

•Dynamics is the volume at which sound is heard.

•Tone color (timbre) is the quality of the music, such as classifying the music as being rich, brassy, bright, warm, harsh, hollow.

•Scales are groups of eight pitches that are harmonious; the diatomic scale is the one we use, and chromatic includes black keys.

•Rhythm is the time of the musical aspects (particular aspects of the arrangement of notes with different durations of time; beat is how music is measured.

•Accent is making certain beats stronger. Meter is an accenting pattern that is repeated (duple, triple, compound); syncopation is flipping the meter such as one TWO to ONE two.

•Tempo is speed (100 -- easy march, 42 -- very slow, 160 -- very fast) or the number of quarter notes per minute. Frequency is the amount of sound waves per second related to pitch.

Fractal music

Other, but definitely not simple ways to find relationships between math and music include excursions into graph theory and randomness. I will make some attempt to explain random music -- or noise, since it is not always of a repeating pattern that is pleasing to the ear -- to be played on a piano.

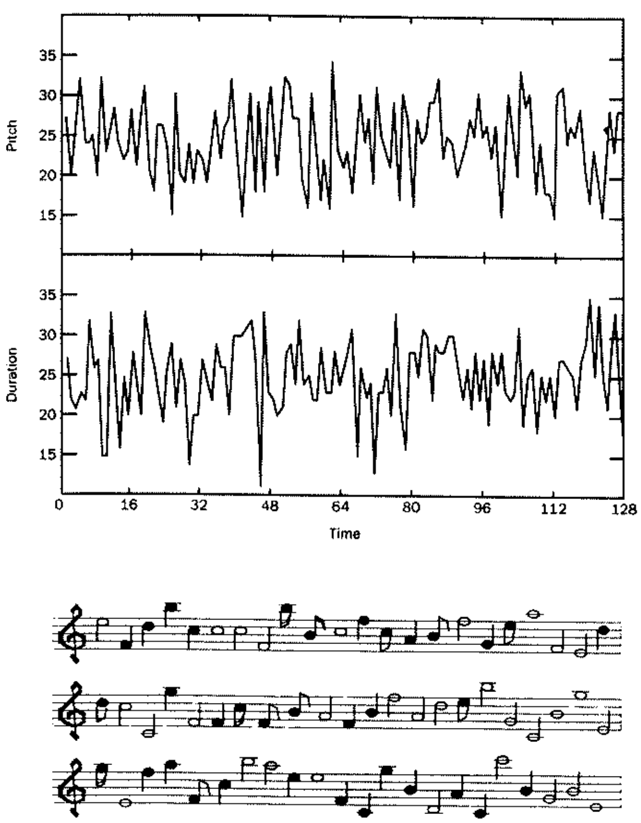
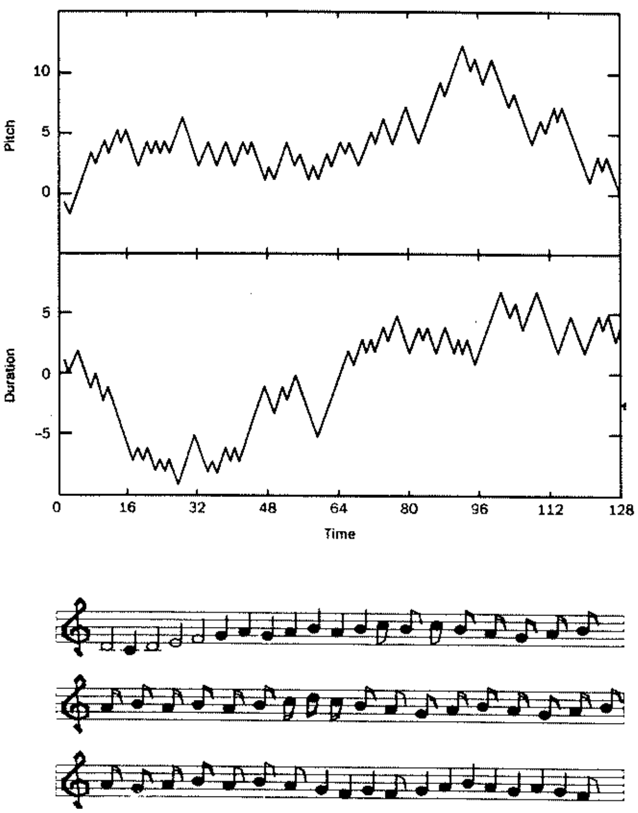


Fig. 3. Duration, Pitch, and Time of White Music. Martin Gardner, Fractal Music, Hypercards, and More... (New York, New York: W.H. Freeman and Company, 1992) 17.

White music is generated in a very simple and random way. If the keys are restricted so as to only involve the eight white keys from middle C to the next C, a spinner can be made with eight portions, which need not be equal, and spun to determine the notes.

Brownian music (or brown music) is, in essence, a two dimensional random walk where the piano's length is considered infinite by looping back to the other side. From a starting point (middle C), a coin is flipped to determine if the next note played should be up the scale or down the scale. Keep in mind that in a two dimensional random walk, the starting place is always returned to (if given an infinite amount of time). Brownian music is more developed then white music and sounds more like parts of music by Mozart or Beethoven.

Even more complex and chaotic, is fractal music or 1/f music. The following algorithm explains how to simulate fractal music: select 16 adjacent notes, write the numbers zero through seven in binary and assign one die each to the one's digit, the two's digit, and the four's digit. Toss all of the dice (making sure to know which is which); the sum of the numbers on the dice corresponds to the next note to be played. Then pick up the one's digit dice and roll it again. Again, the sum is the next note. Now, roll the one's and two's digit dice. In short, every roll involves a change in the one's digit, every other a change in the two's and every fourth in the four's digit. Note that this is the same as the digits that involve a change in counting from zero to seven in binary. If you add another die, of course, you can form a longer sequence encompassing twenty-four tones and giving you sixteen rolls. This ingenious method was discovered by Richard Voss who wrote about 1/f noise in nature and music.[3]



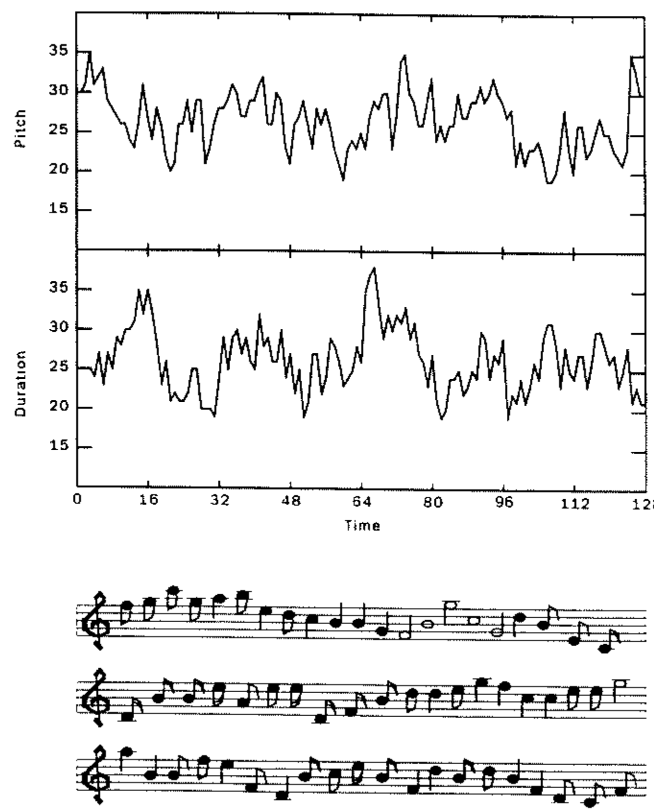


Fig. 4&5. Brownian and Fractal music. Martin Gardner, Fractal Music, Hypercards, and More... (New York, New York: W.H. Freeman and Company, 1992) 18-19.

3 Martin Gardner, Fractal Music, Hypercards, and More... (New York, New York: W.H. Freeman and Company, 1992) 1-22.

Another look at music and math

Among others, the book *Music and the Mind* devotes a very large section to how music affects emotions and how certain pieces were meant to evoke particular feelings. Another approach to this was taken in the MIT math journal which published an article about music and math being closely related within the brain. The results of a survey are shown below.

Response #1	Respondent	Response #2	Respondent	Response #3	Respondent
Computation	a,b,c,e,k,l,m	Patterns	a,g,h,i,m,n	Easy research	a,d,e,h
Visualization	c,d,j,k,l,m	Aesthetics	i,m,n	Creativity	d,i
Patterns	g,h,i,l,m	Creativity	c,d	Background	c,j
Creativity	b,c,e,i,n	Other	a,d,g	Not hard res.	a,d,e,h,i,l
Abstract	g,h	No	b,e,j	No in general	b,f,g,k,m,n
Other	a,f,h,i,j,n	Uncertain	f,l,k		

Above chart is "Mathematicians using music" Leonardo Journal. The letters a through n represent the fourteen different mathematicians who were surveyed. Question number one was "What cognitive skills do you use in your research?" Question number two was "Do you perceive a relationship between music and math?" Question number three was "Do you listen to music while doing mathematical research?"

The musical experience – different views

Regardless of how much one can find to analyze in music and how simply genius certain technical devices are, it is important not to forget to sometimes **listen** to the music. Not to study the techniques, but to just be swept up by the feelings and let the music take you in..

Music, to some mathematicians, is considered a non-serious corruption and a senseless excursion in general number theory. Music, which cannot exist at all without math[4], does not add anything at all to math and actually taints it with a ridiculous, simple, and possibly pleasing and enjoyable form (heaven forbid anyone have enjoyment). Those mathematicians can frown upon persons who consider music to be an art, a math, or simply consider it at all. It is a vulgar form that cheapened formulated math.

[4]Ah, you say, but music does exist without math; it is found in the sounds of nature.

On the complete opposite end, there are those musicians just as un-open-minded.[5] Math, they say, detracts from the music (which is true until you consider that music probably is math and vice versa). There are some pieces, mostly from the 1940's and 50's that take math in music to new extremes. We are no longer discussing tedious computations or music theory guided by math, but the actual preference of incorporating mathematical ideas over the beautiful qualities of the sound. This has at many times, led to music becoming diminished to little more than noise.[6] The music of Shoenberg, Berg, and Webern is as such, where the ideals of twelve tone theory override that of the actual musical experience.

While there is this great conflict between the groups, it has been shown that there are many correlations between musical and mathematical thinking. [7] In nearly every decade of recent history, there has been a great mathematical mind who is also a musical genius. Biographer Davenport says,

> "Until just before his sixth birthday, then, Wolferl led a happy and not too burdened life. ... He learned his lessons, whatever they were, easily and quickly. His mind was usurped by music until he discovered the rudiments of arithmetic. Suddenly the house erupted with figures scribbled on every little bit of space -- walls, floors, tables and chairs. This passion for mathematics is plainly in close alliance with his great contrapuntal facility. Music, however, was his only real interest."[8]

Also take for example, Schoenberg, Bartok, and many mathematicians. Much as Lewis Caroll was essential in the development of literature, Charles Ludwidge Dodson's contributions to mathematics were equally as great.[9] It is the same with many composers and musicians, especially during Romanticism and the last century. To look at a very old example, Perhaps one of the earliest examples of someone with an interest in both math and music was Pythagoras, who was able to appreciate music by considering it one of the maths. [10]

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[5] I am not, regardless of how this sounds, trying to give this a negative connotation. They want to preserve the purity of music, much as mathematicians wanted to preserve the purity of math. Both are stunning in their pure forms and stand perfectly well alone, but I would also like to think that people could look for the compromising blend of the two.

[6] Noise being sound in general, and music being the higher class of noise distinct because of its repetition and pleasing, melodious, and harmonic qualities to listener and the ear. (Note that to look at this from a strictly mathematical perspective, we would say that its graph is regular and periodic; from the musical point, it creates an aesthetic experience.

[7] There are many theories being researched that they even interpreted in the same part of the brain. (The recent Leonardo Journal, for one, elaborates on this idea.)

[8] Wendy S Boettcher., Sabrina S. Hahn, and Gordon L. Shaw, "Mathematics and Music: A Search for Insight into Higher Brain Function," The Leonardo Journal. (M. Davenport Mozart (New York: Avon Books, 1979).

[9] Lewis Carrol is the pseudonym of Charles Ludwidge Dodson, formed by translating his name into Latin and then back again.

[10] Pythagoras was a sixth century B.C. mathematician who's ideas form the basis of scales and tuning in music even today. He will be discussed in great detail later in this report.

These examples lead me to the conclusion that there is some balance between the two extremes. It is much like the parable of one looking for a tree, but seeing only a forest, and another looking for a forest without seeing any trees. What is a forest without trees? Analogously, what is music without math?[11] If the mathematician is looking only at the trees, (s)he will miss seeing the beauty of all of the trees together; if the musician looks only at the forest, (s)he cannot make out the individual trees and will have no idea what that forest really is. There must be some way to look at the trees in the forest and be aware of both the trees and the forest simultaneously (at least to some extent). The above mentioned people were those who were able to consistently tie the two successfully together.

Fig. 6. Vibrations. Theoni Pappas, The Magic of Mathematics: Discovering The Spell of Mathematics (San Carlos, California: WideWorld Publishing/Tetra, 1994) 124.

Sounds, whether in the form of noises or music, are all caused by the vibrations of objects. Once objects, such as a a rubber band, a piece of wood, a wire, or a column of air in a flute begin to vibrate, they cause surrounding molecules of air to vibrate. These vibrations travel outward from the source in three dimensions. When these vibrations reach our eardrums, the eardrums' vibrations send signals to our brains which create the sensation of hearing. Each musical instrument has a method of creating vibrations which in turn cause vibrations throughout the structure and material of the instrument. For example, when a guitar's string is plucked its vibrations cause the other strings and entire instrument to vibrate.

Since ancient times, mathematics has been used to explain music. Today computer modeling and digitization, the quantization of music and sounds coupled with the study of acoustics and acoustical architecture are producing new sounds sensations.

While I have devoted a fairly large section to mathematicians and musicians who fail to recognize the other as something important to the contributions of their scensartsophy[12], I would like to acknowledge that there are many mathemusicians (yes, another dreadful portmanteau) who are able to appreciate math with music and music with math as well as separating the two when necessary.[13] This division between math and music, then, is really no greater or lesser than that between pure and applied mathematicians and musicians who concentrate on aesthetics and those who analyze the music extensively.[14]

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[11]From my own biased standpoint, I refrain from saying what is math without music or trees without a forest, yet I will say, that there is power in number (not number in power which is exponential), that is, one tree alone cannot hold as much meaning as many trees congregated together in one force.

[12]I am using this portmanteau of science, art, and philosophy so as to not label math and music as strictly one or another of these divisions. Perhaps this term will be coined as many of Lewis Caroll's words have been.

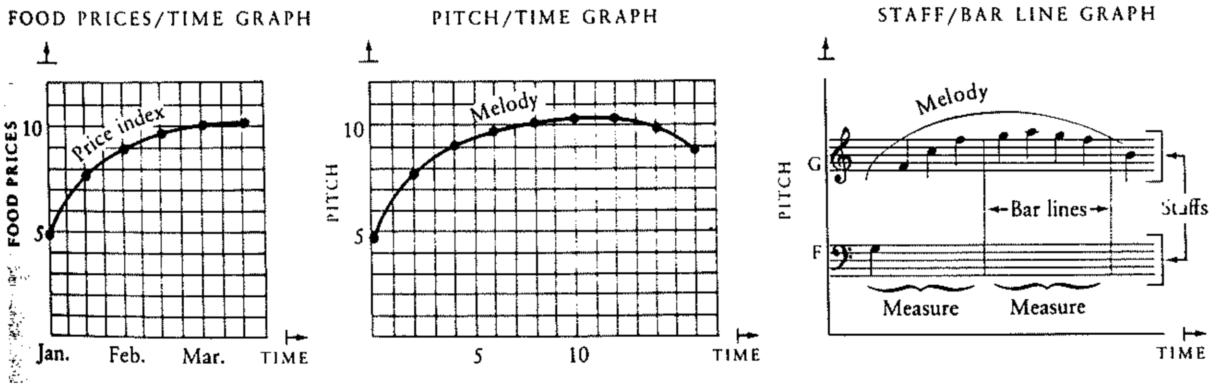
[13]Regretful, I will not go into detail on this; I am already way too far off topic, and so I leave this to the reader to ponder this coexistence.

[14]Pure mathematicians being those who do not "contaminate" math as an art with its scientific applications; applied mathematicians being those who find math unfufilling without its applications to reality (assuming for the moment that reality exists). Again, I will not go into this as much as I would like. (Instead of going off on a TANGENT, I have only briefly TOUCHed on this here.)

One such controversial type of music is that of Schoenberg, which isn't even considered music by some. Because they lack the structure and repetition that we are used to in music, pieces by Schoenberg, for example, sound like little more than noise.

The notation of music theory is highly mathematical in its structure and form. This outline for organization is a scientific approach to studying music. It is interesting to note that aside from the usual way of writing music, many composers of the twentieth century have invented their own, specific for their purposes. Also, graphs can be used to represent musical ideas.

Fig. 7. Musical graphs. Joseph Kerman, Listen: Second Brief Edition (Berkeley, California: Worth Publishers, 1992) 306.



There seem to be two places where it is universally realized that there is a relationship between science and music: in sound itself and in how it affects us. Sound and acoustics are mathematical in two regards. First, the production of sound and the tuning of notes and scales is mathematical, and secondly, the actual auditory reception is scientific. The affects of music, that is, the emotions it triggers within us are caused by what the composer does in writing the piece. Music can affect the brain by arousing certain feelings or stir us by heightening sensitivity.[15]

Music is, therefore, not only a form of enjoyment, but also a deep and psychological body for us to probe at, learn from, and discover from within. If we look past the euphony and sensuousness of a piece, we can find form and some reflection of life staring back at us. Like all other forms of art, music is often used by a composer to convey a certain message.

One of the most interesting books that I have read for this report is Gödel, Escher, and Bach. While the book had little to do with music and trigonometry, it was rich in the striking correlations of the three above-mentioned and Turing, Babbage, Cantor, Golbach, Fermat, Lewis Caroll, and many others as well, and engaged in some humorous word play along the way.

[15]Note: it is a well believed that listening to classical music stimulates both plant growth and psychological and academic growth in the mind.

The picture is by Escher and the score of music below is from Bach's so-called "Crab Canon" in which the piece can be read the same from the top or the bottom. (See below,) Another ingenious idea in music was incorporating the notes B, A, C, and H into The Art of Fugue. [16] The book discusses the differences between Cage and his aleatorical music and Bach, goes into depth on the meaning of fugues and The Well-Tempered Clavier, comparing the Golberg Variations to Goldbach's conjecture, and analyzing a piece that was written for his wife.

Fig. 8. "Crab Canon", *by M. C. Escher (~1965).*

Fig. 8&9. Crabs in Music and Art. Douglas R. Hofstadter, Gödel, Escher, Bach: An Eternal Golden Brain (New York, New York: Vintage Books, 1989) 198, 202.

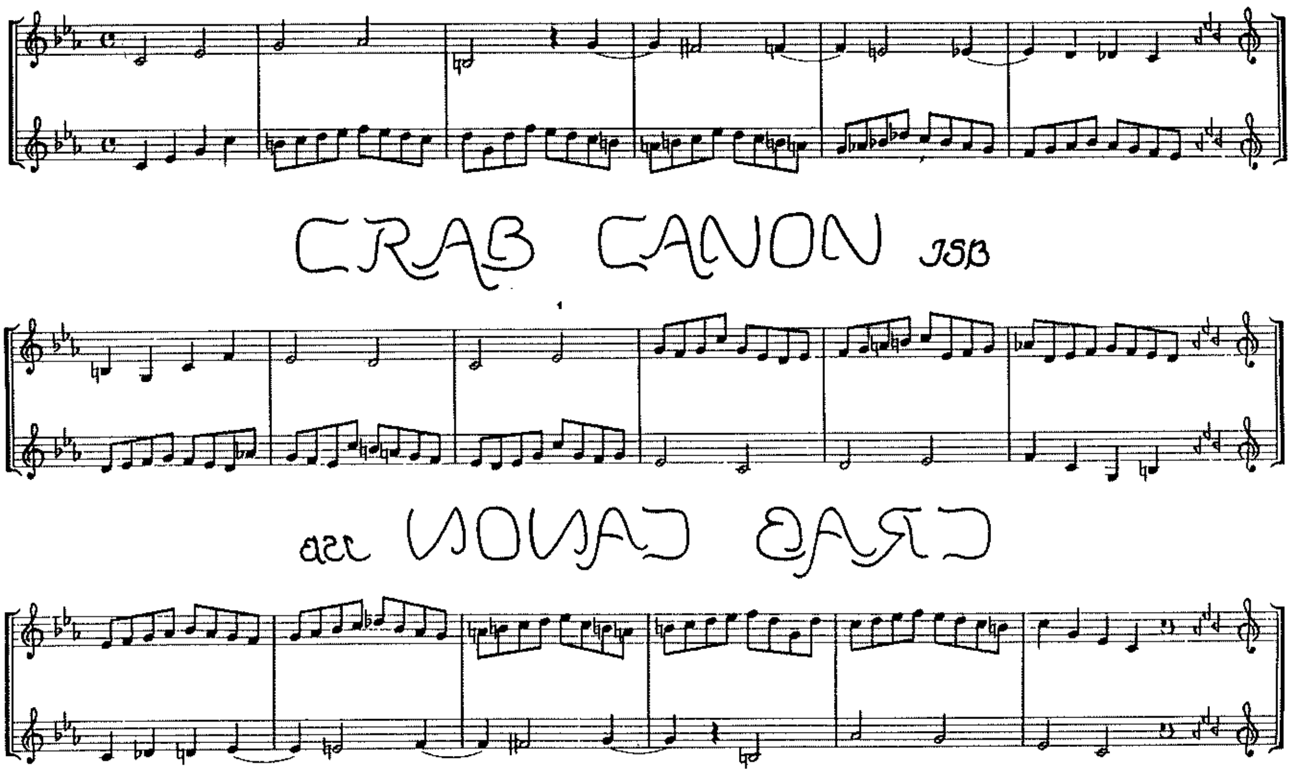


FIGURE 9 Crab Canon *from the* Musical Offering, *by J. S. Bach.* [*Music printed by Donald Byrd's program "SMUT".*]

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[16] H is considered to be a note on the German scale.

Number theory

> "Number theory has long been considered the very paradigm of pure mathematics. Its application to acoustics and to other real-world problems would come as a surprise to many mathematicians more interested in pursuing the properties of integers. Even the term integer, Latin for 'the untouched one,' gives a sense (sic) a being above and beyond the concerns of everyday life."[17]

In architecture and other arts, it has long been known that certain values and ratios are especially pleasing to the mind. One such example is phi (φ), also known as the Golden ratio or Golden section. The ubiquitous phi occurs in the Pyramids and other buildings, many works of art, in astronomy, and in music. It is believed by some that there is a similar pleasingness of phi in music.

Mersenne's Laws (1636): When a string and its tension remain unaltered, but the length is varied, the period of vibration is proportional to the length (Pythagoras). When a string and its length remain unaltered, but the tension is varied, the frequency of vibration is proportional to the square root of the tension. For different strings of the same length and tension, the period of vibration is proportional to the square root of the weight of the string.[18]

Techniques and composers

As many people will tell you if asked (and often if not), math loses much of its meaning if it does not have a purpose. One of the most important things in math is how it relates to our everyday world. As many a music scholar will tell you, there is much to be learned by studying scores of music. There are trends in music that can be followed and used to identify other pieces of the same time period or even by the same composer. One cannot fully appreciate what a composer has done with a piece without knowing what devices they have used: the inversions commonly found in Medieval music, text painting in the Late Renaissance, and the chromaticism of Wagner and others in Romantic opera.

Applying an idee fixe or a leitmotif is one musical procedure that is commonly used to identify a certain character or idea that is re-occurring within a large musical work. By so recognizing these assigned pieces, a certain emotion may be triggered, and one can find subtle hints to what the music is implying. There are many other reasons why such techniques should be used.

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[17] Ivars Peterson, Islands of Truth: A Mathematical Mystery Cruise (New York: W.H. Freeman and Company, 1990) 192.

[18] Sir James. Jeans, Science and Music (New York, New York: Dover Publications, Inc., 1968) 64.

On the two extremes, we have Haydn and Wagner. Haydn was considered to be a "musician's musician" or "connoisseur's composer. Haydn was modest and religious. Although he led a seemingly very dull and regular life and was a person with many obsessions, be maintained a good sense of humor. Unlike Beethoven, who was his student, and other musicians, he gave credit to both his teachers and his pupils. His music evokes deep feelings and relays his inventiveness and originality, as well as his up-beat personality. The music of Wagner, perhaps because he touches on penetrating emotions which people would often rather have left untouched or because of his personality, is disliked and repulsed by many people. Wagner was treated like a god. But, since great music transcends the individual who created it, we can appreciate the power and beauty after we overcome what it is in his music that disturbs us. [19]

Philosophers

Philosophers had much to say about music. "He [Freud] despised music and considered it solely as an intrusion! For that matter the whole Freud family was very unmusical."[20] Nitchze, on the other hand, was a strong supporter of music. He said, "Art and nothing but art! It is the great means of making life possible, the great seduction to life, the great stimulant of life."[21] To a lesser extant, the philosophy of Jung and many other philosophers influenced music, but the philosopher with the most profound effect on a composer was probably Schopenhauer, whose ideas were those on which many of Wagner's works were based.

According to Schopenhauer, all human experience contains both the Will -- emotions and drives -- and the Appearance -- ideas, morals, and reason. The Will dominates the Appearance. Romantic opera and other works played on the conviction that music was the medium to exhibit the dominance of the Will, leading to the many dramatic scenes of opera. [22] Schopenhauer claimed that music is the most expressive of the arts because it does not use Ideas. Our experience of music creates an intimacy which we cannot even begin to comprehend. [23]

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[19] Anthony Storr, Music and the Mind (New York, New York: Ballantine, 1992) 115.

[20] Anthony Storr, 90. (From Harry Freud, 'My Uncle Sigmund' (1956) in Freud As We Knew Him, edited and introduced y Hendrik M. Ruitenbeek (Detroit: Wayne State University Press, 1973) 313.)

[21] Anthony Storr, 150. (Friedrich Neitzsche, The Will to Power, translated by Walter Caufmann and R. J. Hollingdale (London: Weidenfeld and Nicolson, 1968) 452.)

[22] Joseph Kerman, 287.

[23] Anthony Storr, 145.

Beauty in math

Whitehead said, "The science of Pure Mathematics, in its modern developments, may claim to be the most original creation of the human spirit. Another claimant for this position is music."[24] G.H. Hardy said, "A mathematician, like a painter or a poet, is a maker of patterns ... The mathematician's patterns, like the painter's or the poet's, must be beautiful; the ideas, like the colours or the words, must fit together in a harmonious way. Beauty is the first test: there is no permanent place in the world for ugly mathematics."[25]

Beethoven was the bridge between Viennese Classicism and Romanticism in music. Two important themes to Romanticism were the transcendence of artistic barriers and the glorification of music as the most profound of all of the arts. Composers strived to give their pieces some sort of a meaning. In this age lieder (songs) and programme music (music set to a certain literary body but without incorporating the words) flourished. In the 20th century, the avant-garde composers placed a large emphasis on mathematical patterns and devices.

Opinions

As I summed it up in responses: While I consider myself to have more of a tendency toward pure math than to applied math (alas, beauty prevails!), but still I think that there are some things common to music and math and that they are of great importance (surely these relationships are in themselves beautiful). Although I agree that listening and "just" appreciating music is an enjoyable and fully rewarding experience, it is also important to probe deeper at times into the realms of science to discover the techniques that create the beauty. Math and music are both two beautiful and expressive things, each in their own right, and so together they should be able to create a special magic of their own.

I, myself, being somewhere between these distinct preferences understand the applications of math and enjoy learning about how things that we see everyday relate to math. On the other hand, I feel that math (the logic, equations, patterns, and elegance) is just as stunning and beautiful without those applications. The real joys of math are in appreciating the splendor of the thought and the understanding that comes with the solving of a new problem (hence the enjoyment in jumping up and shouting "Eureka!").

Similarly, although I see the importance of learning about the academic devices of a piece, I feel it is just as important (if not more) to be able to listen to the piece, enjoy it, and get something personal out of it.

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[24]Anthony Storr, 178. (Alfred North Whitehead, Science and the Modern World (Cambridge: Cambridge University Press, 1928) 25.)

[25]Anthony Storr, 178 (G. H. Hardy, A Mathematicians Apology (Cambridge: Cambridge University Press, 1940) 24-25.)

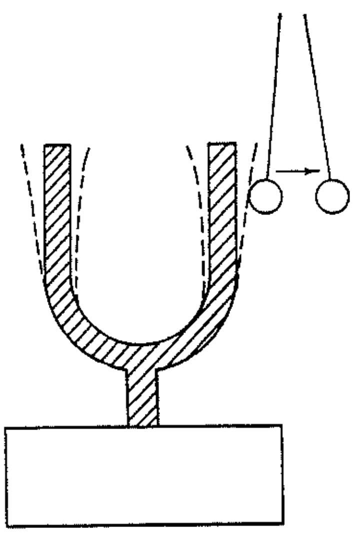
While I did pick the topic applications of trigonometry in music, I also like to be able to enjoy Mozart without analyzing all 626 of his pieces, listen to Pachelbel without calculating ratios of the ground bass (notes that are played over and over that the piece is developed on top of), appreciate Chopin without finding out the exact tempo rubato (literally robbed time) that he used, and enjoy Beethoven just by being moved by the emotion of the early romantics."

This topic aroused some heated discussions, a few such messages of which are included in the appendix.

History of music and math

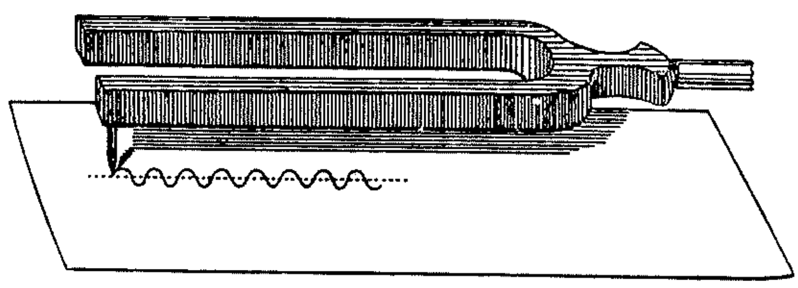
Since early history, two simple instruments have been used. One was the monochord which is a very basic string instrument, and the other is a tuning work, which produces pure tones of frequency by striking it. A monochord is an ancient instrument with a wire at one end fastened to a wooden framework over a bridge. The string is set in motion, and depending on how it is struck and its length, a different sound is produced.[26]

Tuning forks sound much duller than musical instruments because instruments do not produce pure tones. In most instruments, the vibrating of one note causes nearby notes to move as well. If C (64Hz) is struck, c below middle c' (128Hz), G below middle c' (192Hz), middle c' (320Hz), and G above middle c' (384Hz) all vibrate as well.[27] However, in a tuning fork, the prongs are struck and set into motion, they make a sound – the simplest sound of all music.[28]



The vibrations of a tuning-fork give a fuzzy appearance to the prongs and cause them to repel a light pith ball with some violence.

Fig. 10&11. Tuning Forks. Sir James Jeans, Science and Music (New York, New York: Dover Publications, Inc., 1968) 17-18.



The trace of a vibrating fork can be obtained by drawing a piece of paper or smoked glass under it.

[26] Sir James Jeans, 62.
[27] Anthony Storr, 58.
[28] Sir James Jeans, 17.

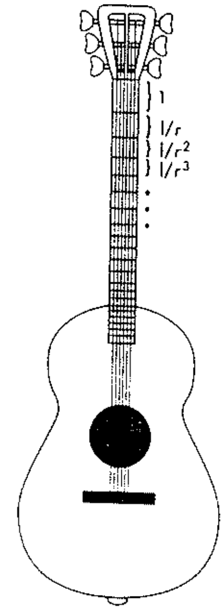


Fig. 12. Ratios of String Lengths. Ian Stewart, Another Fine Math You've Got Me Into.... (New York, New York: W.H. Freeman and Company, 1992) 244.

Pythagoras of Samos, c.560-c.480 BC, was a Greek philosopher who made many developments in astronomy, math, and music. He founded a philosophical and religious school of many followers.[29] The motto of the Pythagorean school was "Number rules the universe". Pythagoras developed the natural harmonic series on which our Western musical system is based.[30] Pythagoras discovered some numerical facts about plucking strings and the ratios of the strings' lengths.[31] Pythagoras discovered the mathematical rationale of musical consonance from the weights of hammers used by smiths.[32] He found "common" musical intervals, that were caused by vibrations and that when the ratio of the lengths were rational they could be used to to produce harmonious tones.[33,34] From C to CC the ratio is 2:1; from C to G the ratio is 3:2; from G to CC it is 4:3.[35] The intervals C D, D E, F G, G A, and A B have a ratio of 9:8. The interval E F and B C have the ratio 256:243. Pythagoreans called this smaller ratio a hemitone, but it is not exactly half of one tone (or my Pentium isn't working; nope, it really isn't).[36]
Instead of scales, there were modes: among them Lydian, Ionian, Phyrgian, and Dorian (these modes were still used during Medieval times). The Pythagoreans applied these ratios to making monochords, and were able to mathematically define much of our musical system. For the Pythagoreans, as well as for Plato, music consequently became a branch of mathematics as well as an art.[37] In astronomy, the Pythagoreans were well aware of the periodic numerical relations of heavenly bodies. The celestial spheres of the planets were thought to produce a harmony called the music of the spheres. Pythagoreans believed that the earth itself was in motion.

[29] "Trigonometry," The New Grolier Multimedia Encyclopedia, 1993 ed.

[30] Theoni Pappas, The Magic of Mathematics: Discovering The Spell of Mathematics (San Carlos, California: WideWorld Publishing/Tetra, 1994), 176.

[31] Joseph Kerman, 10.

[32] "Greek Music," The New Grolier Multimedia Encyclopedia, 1993 ed.

[33] Eli Maor, To Infinity and Beyond: A Cultural History of the Infinite (Princeton, New Jersey: Princeton University Press, 1991) **41-42**.

[34] "Pythagoras," The New Grolier Multimedia Encyclopedia, 1993 ed.

[35] Eli Maor, 41.

[36] Sir James Jeans, 167.

[37] "Greek Music."

Frequencies are highly mathematical. Pythagoras discovered their relationship to exponential equations.

Note	Exponential	Expanded	Note	Exponential	Expanded
C sharp	1.05946	1.05946	G	1.05946^7	1.4983
D	1.05946^2	1.1225	G sharp	1.05946^8	1.5874
D sharp	1.05946^3	1.1892	A	1.05946^9	1.6818
E	1.05946^4	1.2599	A sharp	1.05946^10	1.7818
F	1.05946^5	1.3348	B	1.05946^11	1.8877
F sharp	1.05946^6	1.4142	C'	1.05946^12	2.0000

Above chart "Frequency ratios within the octave" page 25 of Science & Music.

Pythagorean frequency ratio	Pythagorean interval	Equal temperament frequency ratio
C=1.0000	Tone	1.0000
D=9/8=1.1250	Tone	1.1225
E=81/64=1.2656	Hemitone	1.2599
F=4/3=1.3333	Tone	1.3348
G=3/2=1.5000	Tone	1.4983
A=27/16=1.6875	Tone	1.6818
B=243/128=1.8984	Hemitone	1.8877
C=2.0000		2.0000

Above chart "Pythagorean scales and ratios" page 167 Science and Music.

According to the Pythagoreans and Kepler, the unniverse was constructed on the laws of music harmony derived from rational ratios. (Note: the Greek word for rational is *logos*, which is where we get the word logic.) Kepler, who was very Pythagorean himself, thought (incorrectly) that the five regular platonic solids corresponded to the five planets. He also believed that musical harmony ruled the planets and assigned them each a melody in terms of their distance from sun. This is known as music of the spheres.[38]

[38]Eli Maor, 201

Hauer used a system similar to that of Schoenberg, but instead, he based it on series.

"The main purpose of 12-tone composition is: production of coherence through the use of a unifying succession of tones which should function at least like a motive. Thus the organizatorial efficiency of the harmony should be replaced.

It was not my purpose to write dissonant music, but to include dissonance in a logical manner without reference to the treatment of the classics because such a treatment is impossible. I do not know where in the Piano Concerto a tonality is expressed."[39]

Schoenberg spoke of "the emancipation of dissonance", that is, freedom from having to resolve all tensions in a piece. Atonality was first effected by Wagner's technique of chromaticism. Later in the century, a few other times of music prospered: serialism, electronic music, and aleatory music. Serialism used the twelve-tone system, a fixed ordering of row, thematic unity with transformations, rhythmic patterns, and pitches numbered from one to twelve. Electronic music and musique concrete, also flourished around the middle of this century. The chance music (aleatoric music) of Cage, Debussy, Stravinsky, Bartòk, and Schoenberg contributed to developments as well.[40]

There are specific linear ordering on notes of semitonal scale, and so nothing attributes to the sound except that note itself. Mathematically speaking, the different permutations of the notes constitute the ordering of the "rows", "series", or "sets" to create a composition. All notes of a row must occur exactly once before any of them can be used again. Several techniques may be applied to this row: inversion, retrograde, retrograde-inversion, and transposition in Schoenberg's system. (Hauer's system doesn't have retrograde, and Berg's system uses more than one series with little retrograde.)[41] In the atonality of Schoenberg, there is no key, an irregular length, and sprechstimme (an annoying type of speech-voice). From 1914 to 1920, he searched for systematic way to organize, rich varied works, special ordering on tone row, set, or series using every note used once and a tone-color melody.[42]

Fig. 13. Quarter-tone Scale. Kurt Stone, Music Notation in the Twentieth Cent (New York, New York: W.W. Norton & Company, Inc., 1980) 69.

The enharmonics function as follows:

a ¾-sharp C must be notated as a ¼-flat D;
a ¾-flat D must be notated as a ¼-sharp C;

A quarter-tone scale, notated in this system, would read as follows:

Upward

Another, more acurate, scale uses notes between each of the keys on a piano.

Downward

[39] Arnold Schoenberg, Arnold Schoenberg Letters. Ed. Erwin Stein, (New York, New York: St. Martin's Press, 1965), 248.

[40] Joseph Kerman, 341-342.

[41] George Perle, Serial Composition and Atonality: An Introduction to the Music of Schoenberg, Berg, and Webern, (Berkely, California: University of California Press, 1981) 2-5.

[42] Roger Kamien, Music: An Appreciation, (New York, New York: McGraw-Hill Book Company, 1988) 467-468.

The composers of the avant garde are characterized by the twelve-tone, serialism, chance music, minimalist music, quotations of earlier music, electronic music, exploitation of noise, mixed media, rhythm and form.[43] Traditionally an aperiodic form with changing durations, simultaneous motion, and different meters combined without any restrictions was used.[44]

twelve-tone music is a type of Atonal music whereby a chromatic melody, using all twelve tones, is manipulated by the composer in such a way as to avoid any kind of tonality, thereby giving equal importance to all scale degrees. The way this is typically achieved is by manipulating the "tone row" using a number of mathematical formulas.[45]

Computer music uses math in a different way. Composers can use a computer to design sound using formulas for sound waves. Because sound is basically changes in air-pressure, it is possible to create the "blue print" for a sound using a computer and then, through the use of various synthesizers, produce music through amplified speakers. An example of a simple sound that can be produced from an algebraic formula is a sine wave. All the attributes for a given sound, including pitch, duration, volume, timbre, etc., can be designed and stored in the computer using basic computer languages. Because a piece of music must be fully described by the composer using numbers, computer compositions take many hours to produce. One advantage of designing sound on the computer is the composer may produce new sounds.

Some composers that write in this medium are; Bülent Arel (Turkey), Otto Luening (USA) and Milton Babbitt (USA) Princeton Univ. Other composers who may use math to develop musical ideas are; Boulez, Barraque, Carter, Xenakis, Ligeti, Penderecki, Stockhausen, and Cage.[46]

In many of the early synthesizers, and a few more recent models, creation of sounds was based upon the addition of wave patterns to a basic sine wave. The Syndavier II (no longer manufactured) could give you a sine wave of any frequency and would allow you to modify it in an almost infinite variety of ways.[47]

QSound, or three dimensional music, was developed by Danny Lowe and John Lees among others. In QSound, the music seems to be surrounding and coming from everywhere.[48]

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[43] Roger Kamien, 513-515.

[44] Kurt Stone, Music Notation in the Twentieth Century (New York, New York: W.W. Norton & Company, Inc., 1980) 108.

[45] ProfPerc, letter to the author, 12 May 1995.

[46] ProfPerc.

[47] TeacherJRF, letter to the author, 12 May 1995.

[48] Theoni Pappas, 186.

Trigonometric functions

Trigonometry is a branch of mathematics that developed from simple measurement of geometric quantities. Here are some trig tables that I made:

$\sin\alpha$	o/h	y	$\csc\alpha$	h/o	1/y
$\cos\alpha$	a/h	x	$\sec\alpha$	h/a	1/x
$\tan\alpha$	o/a	y/x	$\cot\alpha$	a/o	x/y

The reciprocal relationships of the trig functions	$\csc\theta = 1/\sin\theta$ $\sec\theta = 1/\cos\theta$ $\cot\theta = 1/\tan\theta$
Relationships with negatives	$\sin(-\theta) = -\sin\theta$ $\cos(-\theta) = \cos\theta$ $\tan(-\theta) = -\tan\theta$
Pythagorean relationships	$\sin^2\theta + \cos^2\theta = 1$ $1 + \tan^2\theta = \sec^2\theta$ $1 + \cot^2\theta = \csc^2\theta$
Cofunction relationships	$\sin\theta = \cos(90-\theta)$ $\tan\theta = \cot(90-\theta)$ $\sec\theta = \csc(90-\theta)$

degrees	0	30	45	60	90
radians	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
sine	0	1/2	$\sqrt{2}/2$	$\sqrt{3}/2$	1
cosine	1	$\sqrt{3}/2$	$\sqrt{2}/2$	1/2	0
tangent	0	$1/\sqrt{3}$	1	$\sqrt{3}$	und

quadrants	I	II	III	IV
sin q and csc q	+	+	–	–
cos q and sec q	+	–	–	+
tan q and cot q	+	–	+	–

$$\sin(A+B)=\sin A\cos B+\cos A\sin B$$
$$\sin(A-B)=\sin A\cos B-\cos A\sin B$$
$$\cos(A+B)=\cos A\cos B-\sin A\sin B$$
$$\cos(A-B)=\cos A\cos B+\sin A\sin B$$
$$\tan(A+B)=(\tan A+\tan B)/(1-\tan A\tan B)$$
$$\tan(A-B)=(\tan A-\tan B)/(1+\tan A\tan B)$$
$$\sin x+\sin y=2\sin((x+y)/2)\cos((x-y)/2)$$
$$\sin x-\sin y=2\cos((x+y)/2)\sin((x-y)/2)$$
$$\cos x+\cos y=2\cos((x+y)/2)\cos((x-y)/2)$$
$$\cos x-\cos y=-2\sin((x+y)/2)\sin((x-y)/2)$$
$$\sin 2A=2\sin A\cos A$$
$$\cos 2A=\cos^2 A-\sin^2 A$$
$$=1-2\sin^2 A=2\cos^2 A-1$$
$$\tan 2A=(2\tan A)/(1-\tan^2 A)$$
$$\sin(A/2)=\pm\sqrt{1-\cos A/2}$$
$$\cos(A/2)=\pm\sqrt{1+\cos A/2}$$
$$\tan(A/2)=\pm\sqrt{\frac{1-\cos A}{1+\cos A}}=\frac{\sin A}{1+\cos A}=\frac{1-\cos A}{\sin A}$$
$$\sin(A/2)=\pm\sqrt{1-\cos A/2}$$
$$\cos(A/2)=\pm\sqrt{1+\cos A/2}$$
$$\tan(A/2)=\pm\sqrt{\frac{1-\cos A}{1+\cos A}}=\frac{\sin A}{1+\cos A}=\frac{1-\cos A}{\sin A}$$

Law of Sines: In triangle ABC, $\sin A/a=\sin B/b=\sin C/c$
Law of Cosines: In triangle ABC, $c^2=a^2+b^2-2ab\cos C$

Trigonometric functions have many applications in algebra. They are used in rationalizing square roots. Trigonometric functions are used in polar coordinates, the system in which the position of a point P is determined by its distance OP from a fixed point O and by the angle that OP makes with an initial line OX. In the general application to points in a plane, point O is the pole; OP is the radius vector of P; OX is the polar axis; angle XOP or A, measured counterclockwise, is called the polar angle, vectorial angle, azimuth, or amplitude of P.[49]

[49]"Trigonometry."

Spherical trigonometry involves the concept of a spherical triangle as part of a spherical surface. The shortest distance between two points on a spherical surface is an arc of a great circle. Navigation by sea or air involves choosing routes that are geodesics. The rectangular and spherical coordinates of P are related in the following way: $x = r \sin A \cos B$ $r = \sqrt{x^2 + y^2 + z^2}$ $= r \sin A \sin B = r \cos B$. When r, the radius vector, is constant while B varies from 0 degrees to 360 degrees and A varies from 0 degrees to 180 degrees, the point P generates a spherical surface.[50]

(From discussion on-line about a month and a half ago contributed to mainly by Benjamin J. Tilly, Zdislav V. Kovarik, and Jeff Suzuki)

According to Beckmann's "History of Pi," *sine* comes from the Latin word *sinus*, meaning "cavity or bay", as a result of a medieval Spaniard mistranslating into Latin the Arabic transliteration of a Sanskrit or Greek word meaning "chord". *Tangent* comes from the Latin *tangens*, which means "touching". In the standard picture of a unit circle, the tangent to the circle has a length equal to the tangent of the angle (y/x). Another term from geometry, *secant* comes from Latin again: *secans* means "cutting". Again also, secant is related to the length of the secant of the unit circle being equal to this length (1/x). And the co- in cosine, cosecant, and cotangent are short for complementary.[51]

Math and music time line

•Menelaus (about AD 100) gave spherical trigonometry much attention.

•Ptolemy (about AD 150) introduced an early form of the sine tables in the Algamast as well as an introduction to spherical trigonometry.

•Al-Battani, an Arabic mathematician, produced a table of cotangents and advanced the study of spherical trigonometry, as did Abul-Wafa.[52]

•Leonardo Pisano, (1170 - 1240) also known as Fibonacci, wrote Mis practica geometriae which gave a compilation of the geometry of the time and also introduced some trigonometry.[53]

•Levi ben Gershon, (1288, d. Apr. 20, 1344) a French mathematician and philosopher, wrote an important work about trigonometry.[54]

•Regiomontanus, (1436 - 1476) a German astronomer, played an important role in the revival of Renaissance astronomy and wrote an important work on trigonometry.[55]

50"Spherical Trigonometry," The New Grolier Multimedia Encyclopedia, 1993 ed.

51Benjamin J. Tilly, letter to the author, 8 Apr 1995; Zdislav V. Kovarik, letter to the author, 10 Apr 1995; Jeff Suzuki, letter to the author, 13 May 1995.

52"Egyptian, Babylonian, and Greek Mathematics," The New Grolier Multimedia Encyclopedia, 1993 ed.

53"Leonardo Pisano," The New Grolier Multimedia Encyclopedia, 1993 ed.

54"Levi Ben Gershon," The New Grolier Multimedia Encyclopedia, 1993 ed.

55"Regiomontanus," The New Grolier Multimedia Encyclopedia, 1993 ed.

•Francois Viete, (1540 - 1603) a French mathematician, discovered general solutions to cubic and quartic equations and did important work in trigonometry.[56]

•Simon Stevin, (1548 - 1620) aDutch mathematician and engineer, contributed significantly to the sciences of trigonometry, geography, and fortification.[57]

•John Napier, (1550 - 1617) a Scottish mathematician, was the inventor of logarithms. He also made contributions to spherical trigonometry and found exponential expressions for trigonometric functions.[58]

•Bonaventura Francesco Cavalieri, (1598 - 1647) an Italian mathematician, influenced the acceptance of logarithms and wrote on conic sections, trigonometry, and astronomy.[59]

•Abraham de Moivre, (1667 - 1754) a French mathematician, was a pioneer in probability theory and trigonometry. He devised De Moive's theorem, a trigonometric formula for obtaining powers and roots of complex numbers.[60]

•Peter Gustav Lejeune Dirichlet, (1805 - 1859) a German mathematician, is best known for writing about the conditions for the convergence of trigonometric series used by Fourier in solving differential equations.[61]

•Arnold Schoenberg, (1874 - 1951) an Austrian composer, is famous as the formulator of the twelve-tone system of composition. In 1921 he revealed the "method of composition with twelve tones related only to one another." Schoenberg continued to use the twelve-tone system for the rest of his life.[62]

•Bela Bartok, (1881 - 1945), a Hungarian musician, collected folk music based on the pentatonic scale. One of his pieces isMusic for Strings, Percussion, and Celesta (1936).[63]

•Alban Berg (1885 - 1935), Arnold Schoenberg, and Anton von Webern (1883 - 1945) constituted the so-called Second Viennese school. These men worked together to create one of the 20th century's most far-reaching musical innovations, the method of composition known as the twelve-tone system.

•John Cage, (1912 - 1992) an American composer, rejected the compositional practices of the past to explore a new world of musical sound and structure. He wrote4'33", in which the performer plays nothing for 4 minutes and 33 seconds. Cage sought out the possibilities of chance in music, where music is produced by unpredictable elements.[64]

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56 "Viete, Francois," The New Grolier Multimedia Encyclopedia, 1993 ed.

57 "Stevin, Simon," The New Grolier Multimedia Encyclopedia, 1993 ed.

58 "Napier, John," The New Grolier Multimedia Encyclopedia, 1993 ed.

59 "Cavalieri, Bonaventura," The New Grolier Multimedia Encyclopedia, 1993 ed.

60 "De Moivre, Araham," The New Grolier Multimedia Encyclopedia, 1993 ed.

61 "Dirichlet, Peter Gustav Lejeune," The New Grolier Multimedia Encyclopedia, 1993 ed.

62 "Schoenberg, Arnold," The New Grolier Multimedia Encyclopedia, 1993 ed.

63 "Bartòk, Bela," The New Grolier Multimedia Encyclopedia, 1993 ed.

64 "Cage, John," The New Grolier Multimedia Encyclopedia, 1993 ed.

•Milton Babbitt, (1916+) an American composer, began writing twelve-tone music, but realized that he wanted total organization and control over his compositions. His music does not appeal to the general public, but he would rather be considered a mathematician, or philosopher.[65]

•Yannis Xenakis, (1922+) a Greek composer, began his career as an architect. He invented the idea of stochastic music -- music based on the laws of probability. He uses mathematical formulas to calculate the musical events, which consist of dense sonorities with much sliding from note to note and individual parts for every player in the orchestra. One of his well-known pieces is Metastasis. He explained his theories in a technical book, Formalized Music.[66]

•Gyorgy Ligeti, (1923+) aHungarian composer, is a leader of the European musical avant-garde. He developed Klangflachenkomposition, music constructed with blocks of sound.[67]

•Pierre Boulez (1925+) a French composwer, is known for his carefully crafted, complex compositions. Boulez developed serial music by expanding on dodecaphony to include not only pitch, but also rhythm, dynamics, and tone color.[68]

•Jean Barraque, (1928 - 1973) a French composer, was best known for his serial music. He was a romanticist and a firm opponent of musical collage and aleatory music.[69]

•Karlheinz Stockhausen, (1928+) a German composer, has been a leader in avant-garde music since the mid-1950s. He worked with the experimental composition technique known as musique concrete, in which tape-recorded natural sounds are manipulated.[70]

Scales, vibration, and more

The ratio of the number of vibrations of a plucked string to a string half its length is 1/2. All of the notes in between the two with this ratio form an octave. The concept of the octaves, or some other division of these notes, is an acoustic fact that was not created by man. Octaves are not the only division of notes. Medieval Europe used a system of tetrachords and modes. The Chinese divided the octave into five notes, appropriately called the pentatonic scale, and Indian music is and was improvised by boundaries defined by ragas where the octaves are divided into many intervals called srutis. Persian systems use seventeen or twenty-two notes in an octave to produce even purer sounds.[71]

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[65]"Babbitt, Milton," The New Grolier Multimedia Encyclopedia, 1993 ed.

[66]"Xenakis, Yannis," The New Grolier Multimedia Encyclopedia, 1993 ed.

[67]"Ligeti, Gyorgy," The New Grolier Multimedia Encyclopedia, 1993 ed.

[68]"Boulez, Pierre," The New Grolier Multimedia Encyclopedia, 1993 ed.

[69]"Barraque, Jean," The New Grolier Multimedia Encyclopedia, 1993 ed.

[70]"Stockhausen, Karlheinz," The New Grolier Multimedia Encyclopedia, 1993 ed.

[71]Theoni Pappas, 181.

Celtic folk music is based on a pentatonic system; Hindu music uses half-tones; the slendro (5) and pelog (7) are used in Java.[72] Debussy used the pentatonic scale and the whole-tone scale of six parts. The octatonic scale was also used by Stravinsky.[73] In the sixteen hundreds, Nicolas Mercator proposed a fifty-three note scale, but although the notes are purer, it was extremely impractical to play.[74]

In a letter to the author, Hans Schneider described the relationship between sound scales and mathematics. He said, "You know the tone "A" has 440 Hz (that means cycles per second), and you can find the frequencies of all the other tones by multiplying. Multiplying by the factor f=2^1/12 (about 1.059463) takes you to A#. Another multiplication by this factor f and you have "B", once more and you have "C". Doing this 12 times you will reach the next A one octave higher. This is because (2^1/12)^2=2 and from one octave to the next you have to take the double frequency."[75]

Douglas Zare wrote: "The vibrations of a string, or of air, can be described by the physical forces of the propagation of waves. The solutions of the differential equations which quantify these laws are often best described in terms of trigonometric functions. You might want to look at some books on Fourier series. A guitar string of frequency 200 hertz also vibrates at 400, 600, 800, etc. hertz because these are also solutions of the differential equations which describe waves on a string. The proportions determine why this note sounds like a guitar and not like a flute. If you touch the guitar string lightly (not stopping it) at certain places, you can alter the equations to eliminate some of the solutions. If you do so in the center, then only the frequencies at 400, 800, 1200, etc. will remain, and the sound will be quite different. It will not sound the same as a string plucked normally which is an octave higher. Incidentally, this effect is used by some musicians, especially with electric guitars. Another way that mathematics can help explain music is the concept of consonance and dissonance (?). Basically, two notes are consonant when played together if the ratio of their frequencies can be approximated well by a rational number with denominator. This is related to simple continued fractions, which are introduced in almost any introduction to number theory. Also related is the notion of the difference tone. If I play 100 hertz and you play 102 hertz, we will hear 2 "beats" per second as our waves fall into synch. and out. This is used by musicians to stay in tune while playing. If the difference between the tones is 50 hertz, then we will hear a buzzing as if someone were producing a tone of 50 hertz. If this difference tone is on the scale, and reinforces our notes, then our notes are consonant.

[72] Anthony Storr, 55.

[73] Joseph Kerman, 332.

[74] Sir James Jeans, 189-190.

[75] Hans Schneider, letter to the author, 15 May 1995.

Incidentally, this is why a C-E-G chord is a C chord as opposed to being an E or a G chord: If the C is 200 hertz, then the E is about 250 hertz and the G is about 300. Thus, the difference tones are C's, one and two octaves down."[76]

Dave Buck wrote: The musical scale used most these days is called the "Equal Tempered Scale". This means that each semitone is evenly distributed across the octave - but distributed on an exponential scale. In other words, if you take an A note at 200 Hz and multiply the frequency by the 12th root of 2, you get Bb. Multiply this by the 12th root of two again and you get B. Continue this 12 times and you get to A an octave higher which is 440 Hz. The equal tempered scale allows you to play in any key and it sounds reasonably good. Unfortunately, the equal tempered scale is an approximation, so it's not perfectly tuned. Ideally, a G note would be 1.5 times the frequency of a C note(a perfect fifth) and E note would be 1.25 times the frequency of a C note (a perfect third). In the old days, instruments were tuned this way. The remaining notes in the major scale are derived from perfect fifth's and perfect thirds. Eg. C perfect fifth: G (i.e., 1.5 * frequency of C) C fifth below: F (i.e., 1/1.5 * frequency of C) This is why the chord of C is related to the chords of F and G. G perfect fifth: D (i.e., 1.5 * frequency of G) C perfect third: E (i.e., 1.25 * frequency of C) G perfect third: B (i.e., 1.25 * frequency of G) F perfect third: A (i.e., 1.25 * frequency of F) Put them all together with octaves (1/2 and 2 times each frequency) and you get the perfect scale for C.[77]

Megan Staples wrote: The frequency of middle C is 262. The equation that then fits this is y=3Dsin(524=B9x). The C which 20 an octave above is has a frequency of twice middle Cs or 524. The interval of the fifth (C to G) is produced by a ratio of frequencies of 2:3. Other consonant intervals have other ratios. The amplitude of the function is related to the loudness of the function. Sine waves are only produced by tuning forks and select other devices. In general, the sounds we hear are complex tones with wave shapes that can be mathematically produced by summing sine waves of a variety of amplitudes of frequencies. To created a unified tone of a given frequency- say C 262- all of the partial tones (sine waves) that would be summed together to created that complex tone must have frequencies that are integral multiples of 262. For example, taking 20sin(262*2=B9) + (1/2)sin(524*2=B9)+(1/3) sin(786*2=B9)+(1/4) sin(1048*2 B9)+ C9 20 produces a sawtooth wave is a C.[78]

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[76] Douglas Zare, letter to the author, 17 April 1995.

[77] Dave Buck, letter to the author, 4 April 1995.

[78] Megan Staples, letter to the author, 12 May 1995.

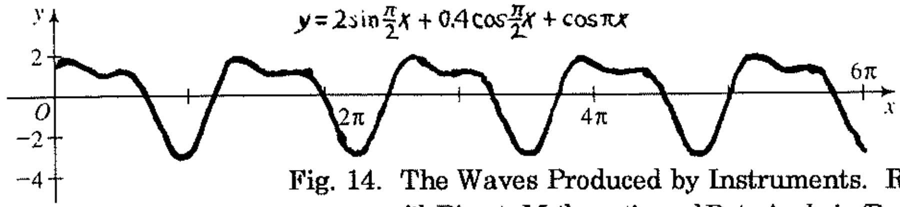
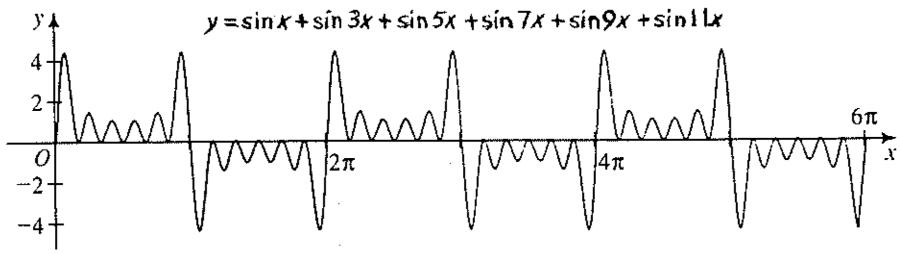
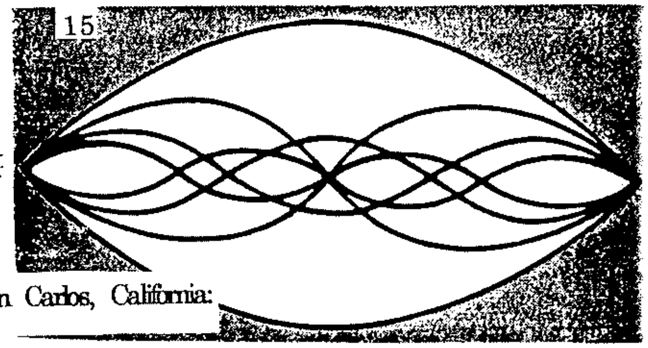


Fig. 14. The Waves Produced by Instruments. Richard G. Brown, Advanced Mathematics: Precalculus with Discrete Mathematics and Data Analysis, (Boston, Massachusetts: Houghton Mifflin, 1992) 380.





Theoni Pappas, The Magic of Mathematics: Discovering The Spell of Mathematics (San Carlos, California: WideWorld Publishing/Tetra, 1994) 178.

The diagram shows a string vibrating in sections and as a whole. The longest vibration determines the pitch and the smaller vibrations produce harmonics.

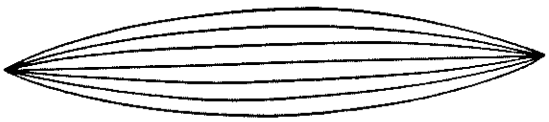
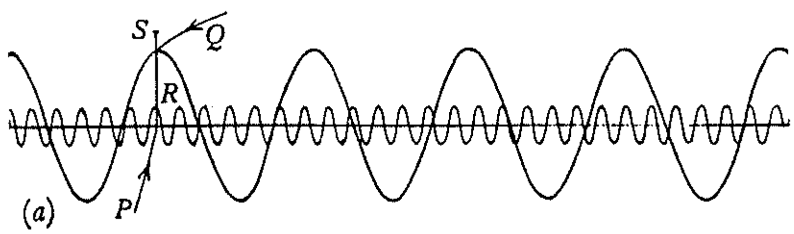
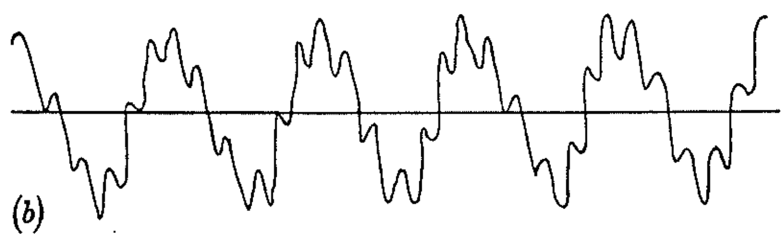
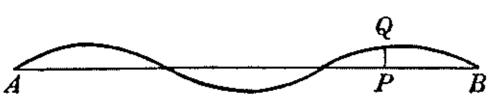


Fig. 16-18. Waves of Frequencies x, 2x, and 3x. Sir James. Jeans, Science and Music (New York, New York: Dover Publications, Inc., 1968) 67.



Sir James. Jeans, Science and Music (New York, New York: Dover Publications, Inc., 1968) 33.





The curve formed by a string performing one vibration alone is always a simple harmonic curve.

Fig. 19. The superposition of two vibrations. The two wavy curves in (a) have periods which stand in the ratio of 6¼ to 1. On superposing them we obtain the curve (b), which represents very closely the sound-curve of a tuning-fork which is sounding its clang tone.

Fig. 20. Sir James. Jeans, Science and Music (New York, New York: Dover Publications, Inc., 1968) 75.

Fig. 21. The response of the ear-drum to a simple sound. The unsymmetrical nature of the human ear results in parts of the simple harmonic curve, which is drawn thick, being replaced by those shewn by broken lines.

Sir James. Jeans, Science and Music (New York, New York: Dover Publications, Inc., 1968) 233.

The pitch of the sound is determined by its frequency--the rate at which the vibrations occur. The units of frequency are called Hertz (Hz); 1 Hz is equal to 1 vibration per second. The human ear is sensitive to frequencies between about 20 Hz and 20,000 Hz.[79]

Tuning forks give perfectly pure pitches. The prongs vibrate when hit and send their vibrations into the air. If a piece of smoked glass were placed under the fork with a needle attached and pulled out at some constant rate, the path traced by the needle would be a sine wave, proving that the fork is vibrating. Sound is, to put it simply, how our ear interprets these vibrations. If many sounds are heard simultaneously, they are superimposed on one another to form a new and more complex curve. Imagine a point P that moves around a circle with uniform speed. Wherever P is, draw perpendiculars to the axes. The points where these two perpendiculars intersect the axes give us harmonic motion. The higher the pitch, the more beats heard per second.[80]
"Music is the pleasure the human soul experiences from counting without being aware it is counting." - Leibniz[81] Sound caused by vibration in all four groups of instruments, though this vibration is caused in different ways. Vibrations travel outward through the air and eventually reach the eardrum where the sine wave is translated into sound.[82]

More relationships

Panels are used, instead of flat surfaces, to reflect and scatter waves to create a richness of sound. This change in architecture of acoustical buildings was brought about by Schroeder. Because the purpose is to scatter sound energy equally in all directions, a larger surface area of wall space at different angles was more efficient than just a flat surface. Related to quadratic-residues mod p. All of this works because stopping waves from meeting wave at crest and reinforcing wave at wave. Strong scatterings found by using primitive roots (closure in group). Both used together for forward-backward and side-to-side.[83]

Fourier analysis is a branch of mathematics that is used to analyze repeating, or periodic, phenomena. Fourier analysis was first developed by Joseph Fourier in the 1820s and has since been highly elaborated.[84] Fourier showed that all sound described by expression of sums of sine functions, and they could be easily graphed in terms of three characteristics: pitch (frequency of the curve),

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79"Sound and Acoustics," The New Grolier Multimedia Encyclopedia, 1993 ed.
80Sir James Jeans, 36.
81Theoni Pappas, 174.
82Theoni Pappas, 174.
83Ivars Peterson, 188-189.
84"Fourier Analysis," The New Grolier Multimedia Encyclopedia, 1993 ed.

loudness (amplitude), and quality (shape). Because of this, computers are able to be used to model sound.[85]

"May not music be described as the Mathematics of sense, and Mathematics as the Music of reason?" - J.J. Sylvester[86] Shapes of instruments, especially strings and the piano often employ special exponential curves to amplify the music, preserve its purity, and be more pleasing to the ear.[87]

A golden section is a line segment that has been divided into two parts in such a way that the ratio of the longer part (a) to the shorter part (b) is equal to the ratio of the entire segment (a + b) to the longer part (a). This can be indicated symbolically as a/b = (a + b)/a = phi (Greek lower-case letter), and this ratio, phi, is called the golden ratio. The concept of a golden section is of historical importance in aesthetics, art, and architecture. It has often been thought that a form, including the human form, is most pleasing when its parts divide it in golden sections. A related concept is the golden rectangle, which the ancient Greeks felt had proportions that were the most aesthetically pleasing of all rectangles; the shape appears in many works from antiquity to the present. It is especially prevalent in Renaissance art and architecture. A golden rectangle has the property such that if a square with side equal to the rectangle's short side is marked off, the remaining figure will be another golden rectangle; this process can be repeated indefinitely. Golden sections also have interesting mathematical properties when certain arithmetic functions are applied. The golden ratio arises in many places in number theory including the ratios between successive numbers in the Fibonacci sequence.[88]

> "Since the early days of Greek art and architecture, man has been aware that beautiful proportions obey fixed rules, and that the most perfect beauty is attained when a great artist's intuition leads him to vary only very slightly something which is mathematically or geometrically perfect. The Greeks devised the geometrical proportion known as the Golden Section which was regarded as the key to the mastery of art for many centuries, and has been abandoned only in more recent times when symmetry and classical proportions are sometimes deliberately shunned. In music, particularly where the composer has adopted a more rational approach, it would seem that some kind of 'Golden Section' has been the foundation of creative thought. Perhaps, subconsciously, man needs to work in orderly patterns or proportions, and perhaps in sound as well as vision these proportions are determining factors in the creation of perfect beauty.
>
> The use of a 'Golden Section' is not limited to composers of the integral serialism period. As long ago as 1925, Alban Berg constructed his Lyric c Suite around the composer's 'magic' number 23, a number which determine the length of movements, metronome markings, and

[85]Theoni Pappas, 186.
[86]Theoni Pappas, 175.
[87]Theoni Pappas, 177-178.
[88]"Golden Ratio," The New Grolier Multimedia Encyclopedia, 1993 ed.

even at some points the number of notes in chords. Oliver Messianen's Technique de mon langage musical reveals a 'marked predilection for the rhythms of prime number'."[89]

Some composers and pieces to look for:
John Cage, Stockhausen, Xenakis (has Pascal's triangle), Koenig (binomial coefficient). Nono (Fibonacci sequence), Xenakis[90]

The form of Bartok's "Music for Strings, Percussion and Celesta," follows Fibonacci ratios and the golden section (root 5 plus one all over 2, or .618...). Also check out Varese's "Poeme Electronique" Many Stockhausen pieces are based on non-traditional scales that are derived through some mathematical procedure. Harry Partch is another composer who creates different scales, as well as his own instruments.[91] Also, Charles Wuorinen's piece "The Golden Dance" is based on the golden section. Wuorinen also was influenced by Benoit Mandelbrot.[92]

I think that I have explored enough aspects of music and its strong relationships to mathematics. Clearly there are many levels to music; the composing techniques and the systems we use for music, psychological aspects, and the use of number theory in music all lend to the beauty of music. Perhaps now when music is played, we can truly appreciate both the skills of the composer and the beauty of the music.

89The New Music, 47-48

90Dana Albrecht, letter to the author, 17 April 1995 (quoted from Vance Maverick, 20 January 1995).

91David Heetderks, letter to the author, 12 May 1995.

92Howard Stokar, letter to the author, 12 May 1995.

Appendix: on-line discussion

The discussion on sci.math has touched on many aspects of music and math including frequencies, vibrations, and tuning, music appreciation, and dissonance and incorrect music. Because the debates of appreciation played an important part in setting the tone of this report, I am including four of the messages on this topic.

From: abian@iastate.edu (Alexander Abian) Date: 16 Apr 95 01:11:39 GMT

The relationship of Math and Music is in the Elegance of the Solution of problems be it a mathematical problem or a musical problem - they both are emotional problems - please do not bring Trigonometric functions, Fourier series, Harmonic analysis, fractions, etc as examples of significant items which supposedly are common in math and music because in music they count the frequencies and they hear the harmonics. Please! The exquisite taste, eloquence and, yes, I repeat the elegance of solution of challenging problems those are the common items relating math to music. Kreutzer Sonata of Beethoven, Schubert's songs, Mozart's and Beethoven's trios and quartet etc, etc, etc., have the same quality of impeccability of taste, elegance, eloquence of reasoning and thoughts as many, many examples in mathematics do, say, dealing with passages from finite to infinite, with passages in real and complex analysis, in algebra, in topology, etc, etc - I repeat again the elegance of solutions of problems whether sensual or cerebral (both being emotional) that is what the common bond of math and music is. In mathematics there is as much sensuality as in music. One may be emotionally aroused, weep, cry and have shivers passing through the spine while following the proof of a theorem just as when listening to the Rasumovski's quartets of Beethoven or to B minor Mass of Bach or to Schubert's quintet in C major or his quintet in A major (the Trout).

From: snowe@rain.org (Laura Helen) Date: 16 Apr 1995 08:53:53 -0700

Replying to Abian: But, taste, eloquence and elegance play a role in other arts too. Think of a great poem, like Wordsworth's "Intimations of Immortality from Recollections of Early Childhood", or a great painting. But mathematical talent is traditionally associated with musical talent in particular. This association may be over-emphasized; there certainly have been plenty of mathematicians who were writers as well. Perhaps some of this has to do with mathematics being creation *within* a definite system of rules; music is also. Perhaps to a somewhat lesser extent so is poetry. By the way, I wonder if this math/music association applies to, say, heavy metal music as well. Are mathematicians likely to be Led Zep fans? Or to do jazz improvisation?

From: pmclean@virgo.drao.nrc.ca (Philippe McLean) Date: 16 Apr 1995 20:06:51 GMT

In response to Laura: Of course. You could as easily ask, are combinatorics people likely to read Woodsworth? Or listen to Bartok? And do algebraists crave Rage against the machine? And aren't all statisticians big on C&W? Taste in music is influenced by culture and generation. So are efforts to characterize good music, for example the previous discussion of ``taste and elegance' and ``beauty'. Music has a universal appeal that eludes crystallization. Music has a great deal of mathematical structure. Thinking about scale, keys, and the division of time is like doing mathematics. There is also the appeal of western musical notation; it is an efficient and precise symbolic vehicle. Structure aside, there is the appeal of idea. Music and math are both full of cool ideas that make you say, "Wow, wicked. Let's work with that."

From: zare@cco.caltech.edu Date: 17 Apr 1995 08:30 AST

Anyway, I have not seen any serious posts regarding your request. Unfortunately the tone was set by Abian, who is believed by many (including me) to be a crackpot. Others seemed to follow on this whimsical level, but there are many serious connections between music and mathematics.

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