Conic Sections, Dandelin Spheres, and Other Interesting Math

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A math project
Mr. Button

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Special thanks to *Her Majesty the Yellow Pig* for inspiration, my parents for dealing with me while I was writing this paper, and to Apollonius of Perga (even though he had to write his book in Greek).

Table of Conics

Conic Sections

Introduction:

A conic section is a curve formed by the intersection of a plane with a right circular cone. A conic section can be generated by a point that moves so that the ratio of its distance from a fixed point (called a focus) to its distance from a fixed line (called a directrix) is constant. In this way conics are defined by their eccentricity (e). (If e = 1, the conic is a parabola; for e < 1, it is an ellipse; and for e > 1, it is a hyperbola.)

Conics can also be generated algebraically by second-degree equations in two variables. Each equation of the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where not all of A, B, and C are zero, generates a conic or a degenerate conic. Using the determinate $B^2 - 4AC$ the type of conic can be determined.[1]

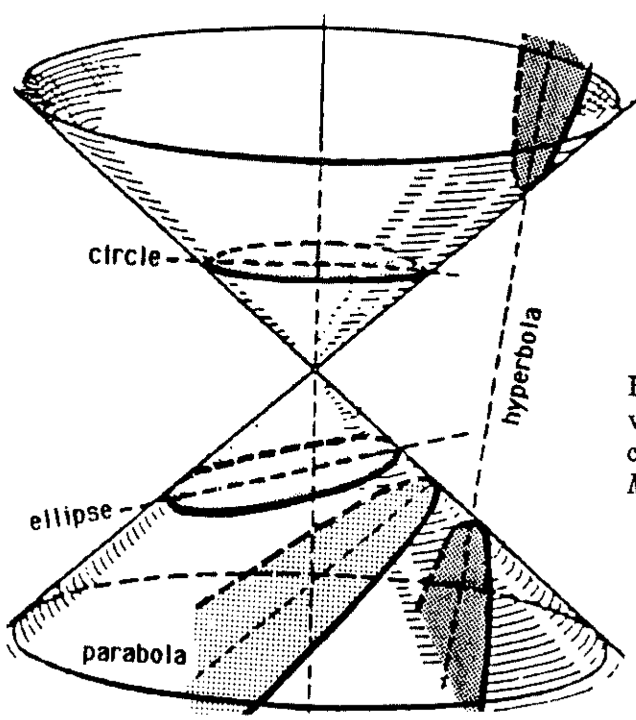


Fig. 1. The intersections of various planes and a right circular cone. (*The Joy of Mathematics,* 197)

An ellipse is a curve formed by a plane that intersects the axis of a circular cone and is not parallel to an element of the cone. An ellipse may also be defined as the locus or set of points for which the sum of the distances to two fixed points is constant. Each fixed point is called a focus. Oblique cross sections of right circular cylinders are ellipses. The focal property of an ellipse gives it special optical and acoustical properties. A light beam or sound beam emanating from one focal point in any direction will always pass through the other focus after being reflected from the ellipse. A circle is a special type of ellipse.[2]

An ellipsoid is a closed central surface that can be represented on three-dimensional, x-y-z-coordinate axes by an equation of the general form $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. When the constants a, b, and c are such that $a > b > c > 0$, the major axis (of length 2a) is along the x-axis, the mean axis (of length 2b) is along the y-axis, the minor axis (of length 2c) is along the z-axis, and the center is at the origin. If $a = b = c$, the ellipsoid is a sphere. If two of the constants are equal, the surface is a spheroid, or ellipsoid of revolution. Rotating the ellipse $x^2/a^2 = z^2/c^2 = 1$ about its major axis produces a prolate spheroid; rotating the same ellipse about its minor axis produces an oblate spheroid.[3]

A hyperbola is formed by a plane that intersects a right circular conical surface and is parallel to the axis of the surface. A hyperbola is the set of points with a constant difference of the distances.

A one-sheeted hyperboloid of rotation, obtained by rotating the two branches of a hyperbola about its conjugate axis, is a connected surface. A two-sheeted hyperboloid of revolution, obtained by revolving the two branches of a hyperbola about its transverse axis, consists of two equal parts situated symmetrically on either side of a plane of symmetry.[4]

A parabola is formed by a plane that intersects a right circular cone and is parallel to one of its elements. A parabola may also be defined as the set of all points that are equidistant from a fixed point (the focus) and a fixed line (the directix). By using this definition, a parabola may be constructed by ruler and compass. The parabola has a very special geometric property that makes it useful as a reflector of light and sound. Light rays passing through the focus and being reflected from the parabola will emerge in a direction parallel to the axis of the parabola; conversely, rays approaching the parabola parallel to the axis will be reflected so that they pass through the focus.

When a parabola is rotated about its axis, it generates a surface called a paraboloid of revolution.[5]

History:

Apollonius of Perga, a Greek mathematician of the 3rd and early 2nd centuries BC, was known as the Great Geometer. In *On Conic Sections*, he introduced the terms *ellipse*, *hyperbola*, and *parabola*. He was also an important founder of Greek mathematical astronomy.[6]

The invention of analytic geometry is generally credited to the French philosopher and mathematician Rene Descartes (1595-1650).[7] Gerard Desargues (1591-1661) was a French mathematician whose work centered on the theories of conic sections, perspective, and projective geometry.[8]

Who is Dandelin?

Germinal Pierre Dandelin was born in 1794 and died in 1847. In 1822 he proved that there always exist spheres in the cone such that they are tangent to both the conic and the cone.[9] These spheres are known as Dandelin spheres. Dandelin was a Belgian mathematician and geometer, as well as a professor of mechanics at Liege University.[10] Now that you are hopefully interested enough to learn about Dandelin spheres, read on. (And even if you aren't interested, still read on.)

What are Dandelin Spheres?

Luckily for me, I was able to get a photocopy of a chapter of Lyle E. Mehlenbacher's *Foundations of Modern Mathematics* which defines conics in terms of Dandelin spheres. Refer to the figures as you read this section (otherwise it will be a miracle if you can understand what I wrote). Also included at the end of this section are two demonstrations that John Conway e-mailed to a friend of a friend.

A Dandelin sphere is a sphere inside the cone so that it is tangent to the cone at the points of a circle. If a plane intersects the cone perpendicular to the axis, forming another circle, then there are two Dandelin spheres in the cone that are also tangent to the plane containing the circle at the center of the circle (one on top, the other on the bottom). Similarly, if you have an ellipse or hyperbola formed by the intersection of a plane and the cone, there will be a spheres tangent to the plane containing the conic at each of the conic's foci. A parabola is slightly different, having only one Dandelin sphere. A Dandelin sphere is tangent to the plane containing the parabola at its focus. Directrices of conics can also be found using Dandelin spheres. The directrices are the lines in which the cutting plane meets the planes of the circles in which the spheres touch the cone.

On Conics and Spheres

Let's first consider the case of the Dandelin spheres in a cone containing a circle as mentioned above. There are two spheres (call them D and E) tangent to the cone and to the circle at its center. Let P be a point on circle C and VP (V is the vertex of the cone) is on the cone and contains P, then VP is tangent to the Dandelin sphere D at the point B, and VP is tangent to the Dandelin sphere E at point A. PO is tangent to each sphere at O (got all of that?), so PO = PB = PA. (A and B are on circles G and H.) Sphere D is tangent to the cone at every point on the circle with center G, and sphere E is tangent to the cone at every point on the circle with center H.

Both circles G an H are on planes that are perpendicular to the axis of the cone (and therefore parallel to the cone's base). Replacing the slicing plane with the cartesian plane allows us to look at all of the conics analytically. Proofs to find the algebraic definitions of conics are based on this principle. (In other words, conics are often defined in terms of Dandelin spheres as opposed to defining Dandelin spheres in terms of conics.)

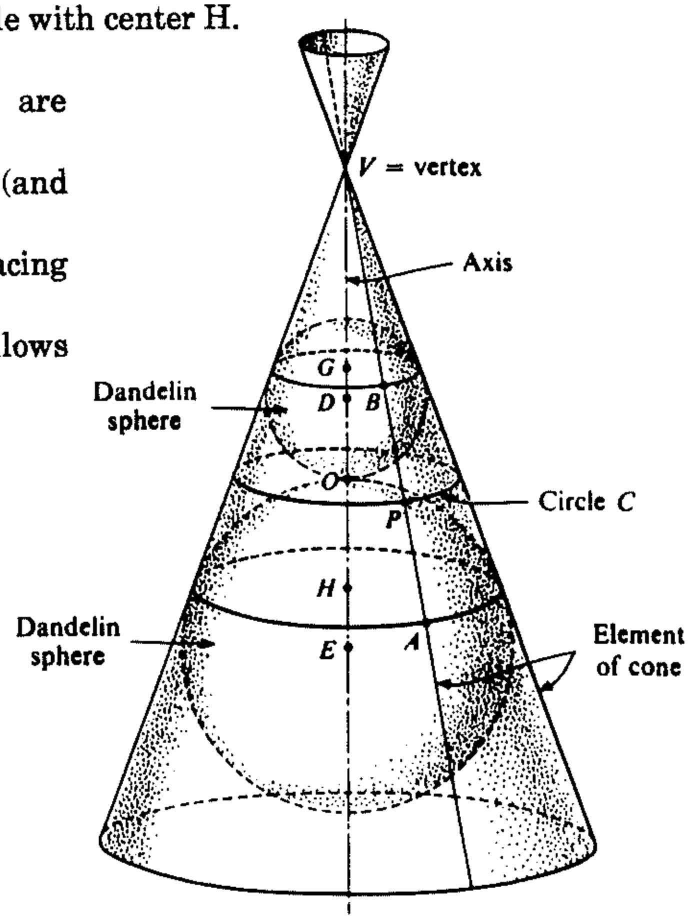


Fig. 2. A circle and its Dandelin spheres. (*Foundations of Modern Mathematics*, 120)

Now let's consider a more general case, the ellipse. When a cone is sliced by a plane, not through V, and so that the intersecting conic lies in only one half of the cone, the conic is an ellipse. One of the spheres is tangent to the cone in circle U and to the slicing plane at the point F_1 (plane through the first focus), the other sphere is tangent to the cone in circle W and to the plane through other focus, F_2. We pick a point P on the ellipse lying on VP. VP is tangent to the Dandelin sphere S at A, where it touches circle U. It is tangent to sphere T at B where it touches circle W. The plane containing circle U is parallel to the plane with circle W. $PF_1 + PF_2 = PA + PB = AP + PB = AP$. The length of AB is constant regardless on which point P on the ellipse. Lines d_1 and d_2 in the picture are the intersection of the slicing plane and the plane of circle U and the intersection of the slicing plane and the plane with circle W respectively. They are the directrices of the ellipse.

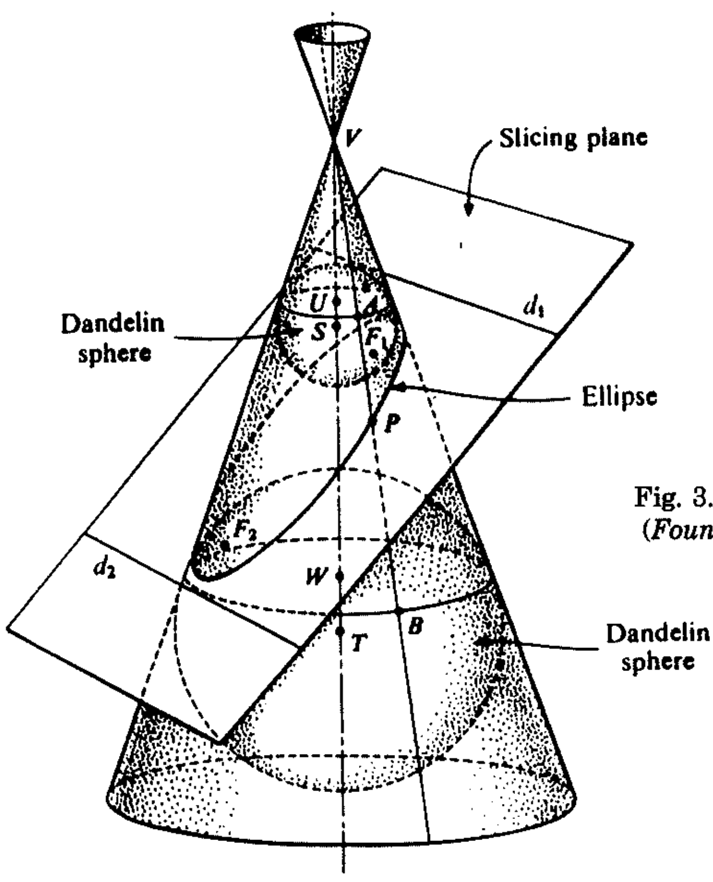


Fig. 3. An ellipse and its Dandelin spheres. (*Foundations of Modern Mathematics,* 126)

Hyperbolas are very closely related to ellipses, so there are many similarities between Dandelin spheres in a cone with an ellipse and one with a hyperbola. In a hyperbola, one of the Dandelin spheres is in the upper half of the cone (sphere S). It is tangent to the cone in the circle with center U and to the slicing plane at F_1. The second sphere (sphere T) lies in the lower half of the cone and is tangent to the cone in the circle with center W and to plane at F_2. Select a point P on one branch of the hyperbola. VP is tangent to sphere S at point A and to sphere T at point B.

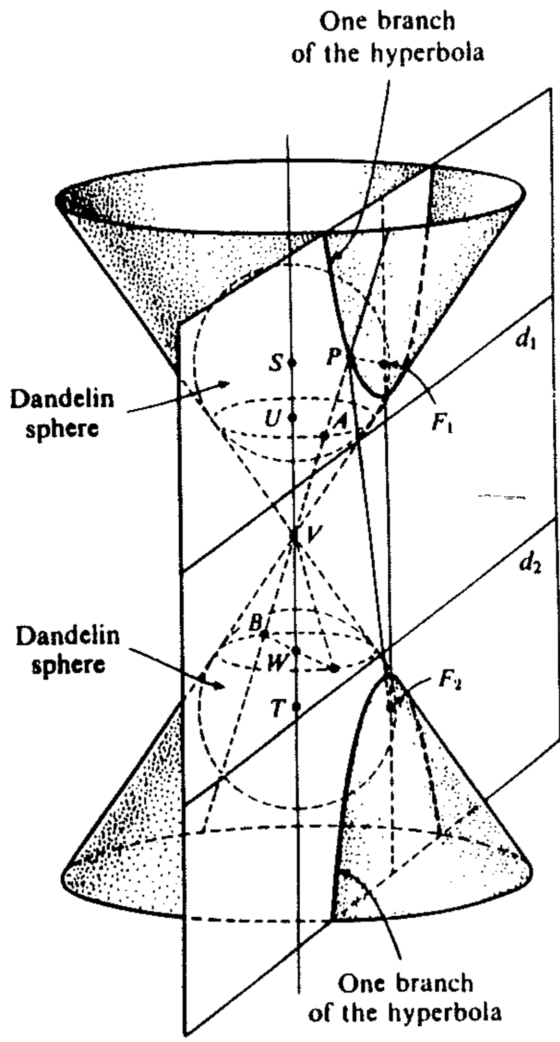


Fig. 4. A hyperbola and its Dandelin spheres. (*Foundations of Modern Mathematics,* 129)

Since P is on the slicing plane, PF_1 is tangent to S at F_1, and PF_2 is tangent to T at F_2. Again, the planes with circles U and W are parallel. $PF_1 = PA$, $PF_2 = PB$, $PB - PA = PF_2 - PF_1 = AB$. Again, d_1 and d_2 are the directrices.

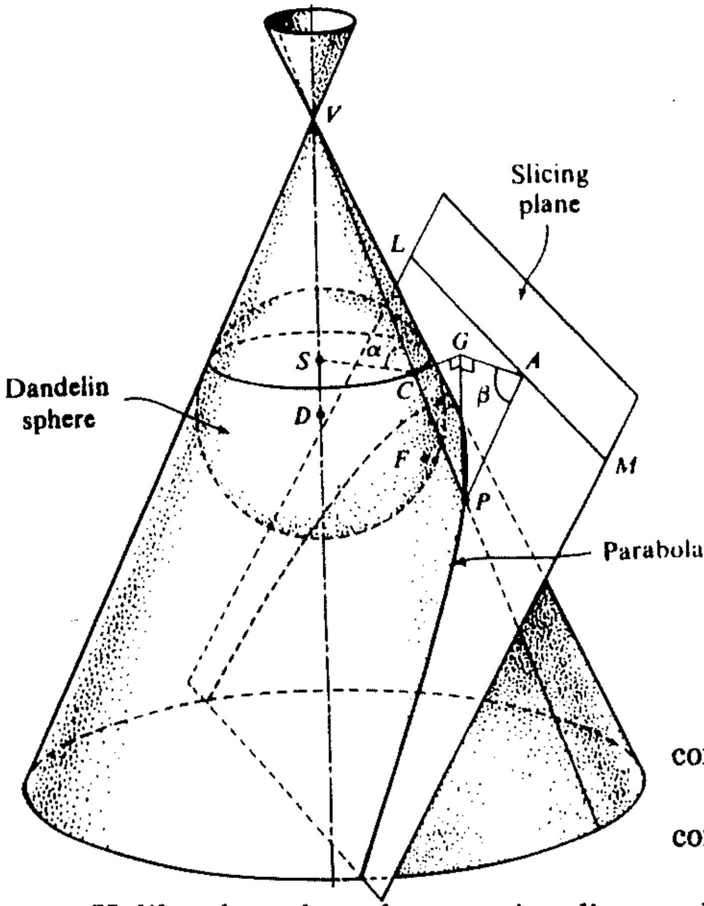
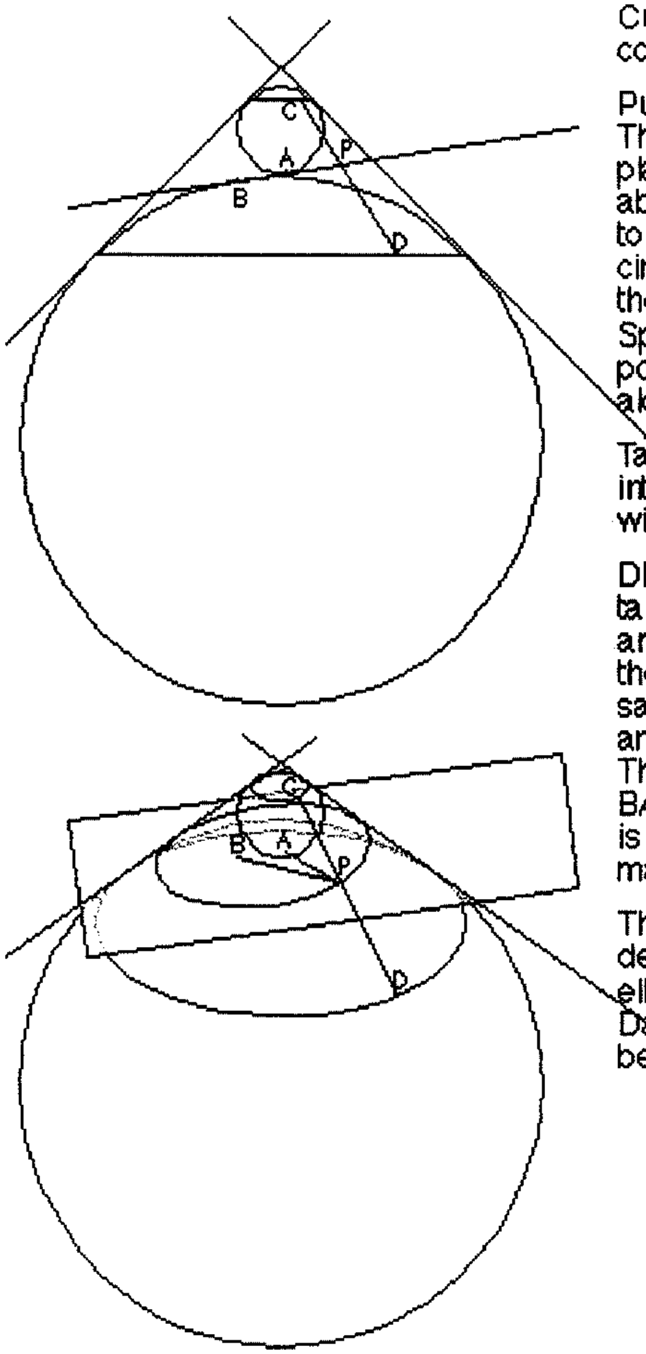


Fig. 5. A parabola and its Dandelin spheres. (*Foundations of Modern Mathematics,* 129)

When the plane that slices a cone is parallel to the cone, then the conic section formed is a parabola. Unlike the other three conics discussed, the parabola has only one conic section tangent to it because its geometric definition involves a directrix instead of a second focus. D, our hero the Dandelin sphere, is tangent to the cone in circle S and tangent to the slicing plane at F. Again, we pick a point P on VP and the parabola. PF is tangent to D at F, and PV is tangent to D at C, so PF = PC. LM is the intersection of the slicing plane and the plane with circle S, so LM is the directrix of parabola. Also unlike the Dandelin definitions of the other three conics, the definition of the parabola is obtained by constructing perpendicular planes to the sphere and then using ideas of triangle congruence. (In the picture, PA is perpendicular to LM at point A. PG is perpendicular to the plane containing circle S. The triangle PGC has a right angle at G as does triangle PGA.) Because the slicing plane is parallel to the edge of the cone, we can conclude that PA = PC = PF.[11]

Here are two Dandelin sphere demonstrations that I received from an on-line correspondent who was given them by John Conway.[12]

Fig. 6. Dandelin demonstration of an ellipse.



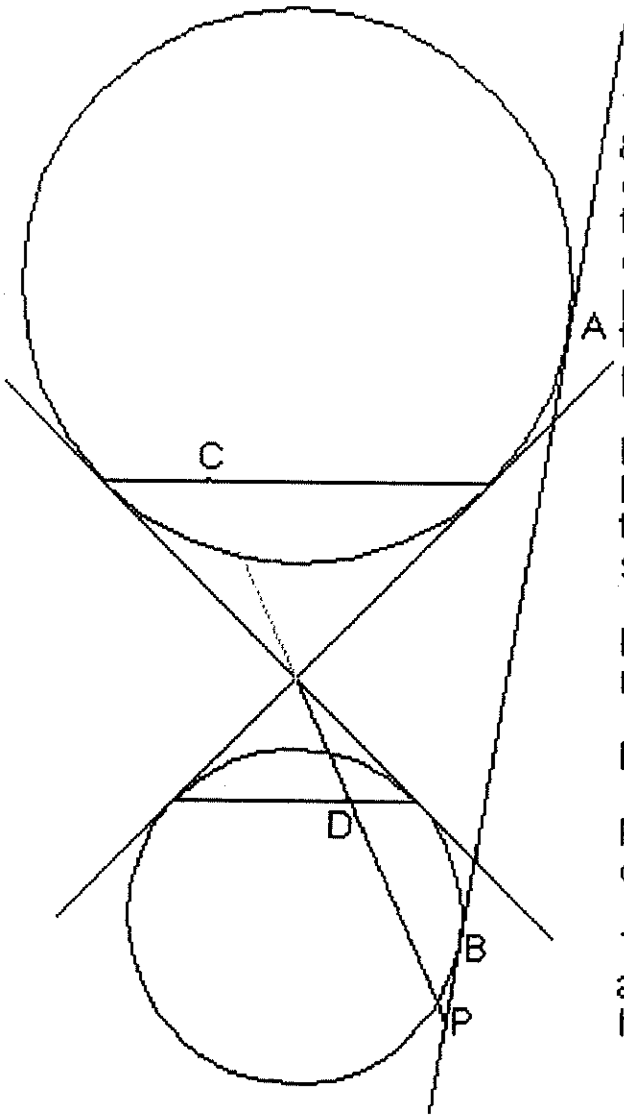
Cut the bottom half of the cone with a plane.

Put in two spheres:
The first touching the plane at a point from above (at point A) and touching the cone along a circle.
the 2nd touching the Sphere from below (at point B) and the cone along a circle.

Take a point lying on the intersection of the cone with the plane, call it P

DP and BP are both tangent lines to the circle and they intersect so they're equal. For the same reason CP and AP are equal to each other. Therefore BA+AP=CP+PD, and CPD is of constant length no matter where P is.

This is the dandelin demonstration for the ellipse. The spheres are Dandelin spheres, I believe.



The plane cuts both the top & bottom halves of the cone. Place a sphere in the top half that touches the cone along a circle and the plane at point A. Ditto for the bottom half touching the plane at point B.

In this case PD=PB because they're both tangents of the same sphere meeting at P.

PC=PA for the same reason.

PA-PB=PC-PD

PC-PD=CD which is constant.

The same argument can be applied to points on the top half.

Fig. 7. Dandelin demonstration of a hyperbola.

Here comes the conic relief. Well, almost. It's still about conics, but it's more relieving, or at least:

Interesting.... I think

As I was searching, reading various books and spending time on-line, I came across some of what I consider to be the most fun of this project -- the little tidbits (that relate to absolutely nothing). Here are a handful of facts (don't expect any of these topics to come up at your dinner table, unless you eat dinner with very, very strange people).

Uses of Conics and Conicoids:

One of the most interesting things I found (the main rival for topic choices) was uses of conics and conicoids. Parabolas have the property that all rays coming in will be reflected and pass through the focus.

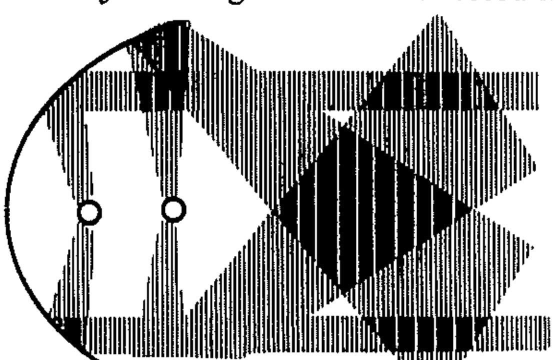


Fig. 8. Parabolas in headlights.
(*The Magic of Mathematics*, 13)

Parabolic mirrors are used in headlights, searchlights, antennas, radar, and telescopes. Paraboloids are also used in speakers.[13] Reflectors behind headlights are paraboloids. A beam is created at the focus so light travels out parallel to the axis of symmetry. Dimming the light changes the focus so most of the light is blocked.[14] Sounding off -- St. Peter's Cathedral in Rome and the Greek amphitheater at Epidaurus use parabolic reflectors for acoustics.[15] (Sounds like a good idea to me.)

Parabolas are also used to approach irrational numbers to solve functions.[16]

Hyperbolas are used in navigation. During World War II a hyperbolic system was developed using U.S. long range navigation, or loran.[17] Loran uses the property that all sounds at one focus will pass through the other focus. The navigator notes the time of arrival of the various signals and can pinpoint the aircraft's position as the intersection of two hyperbolic curves, or loran lines.[18] Flood lights are also hyperbolic. Hyperbolic mirrors make objects look as if they have been reflected from the other focus.[19]

Ellipses are used in engineering, in arches of bridges, and gears for machinery (punch presses).[20] Ellipsoids are used in computers to view higher dimensions (very much higher, like 17 dimensional hYPerspace).[21]

The elliptic curve y^2=x^3+ax+b is used in an algorithm for factoring very large numbers.[22] Ellipses are also used efficiently in bicycle gears.[23] Machines shaped like ellipsoids are used in medicine to pulverize kidney stones without surgery. The machine is placed against the body so that one focus is inside the machine and the other is inside the body at the site of the stone. Because of the property that ellipses will reflect anything from one focus to the other, sound waves that are emitted bounce off of the inside of the ellipsoid and collect at the stone to pulverize it.[24] And another use of ellipses (although not as scientific) -- ever play pool on an elliptical table?[25]

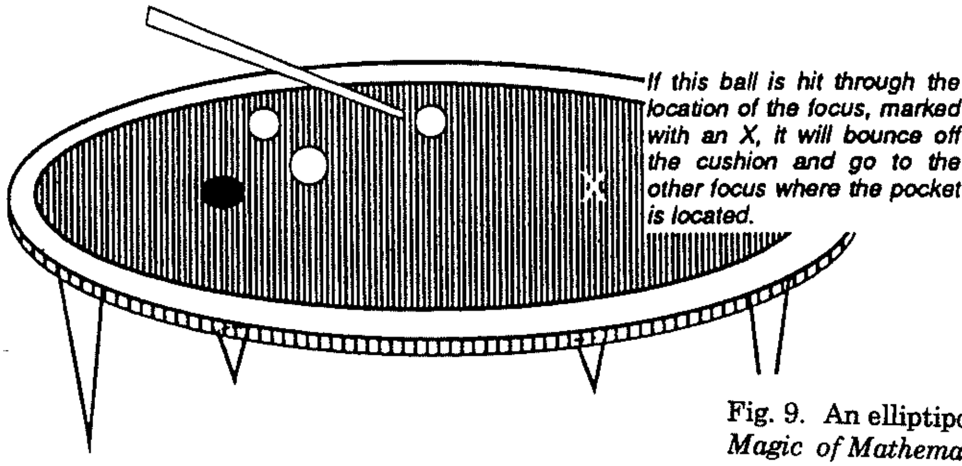


Fig. 9. An elliptipool table. (*The Magic of Mathematics*, 18)

Conics in Nature:

All four conic sections were found by Kepler in the paths of planets. Galileo discovered the relationships that a parabola has to trajectiles. An arc of spouting water and the shape of flashlight on a flat surface are both examples of a parabola. Any rotating liquid in a circular bowl will trace out a parabolic shape. Ellipses and hyperbolas are found in the orbits of comets and planets. Circles are at work whenever we see ripples on a pond, the wheel, or some orbits. Several laws of nature involve hyperbolic curves. Many comet trajectories and paths of particles in atomic-scattering experiments are hyperbolas.[26]

Conics in Buildings:

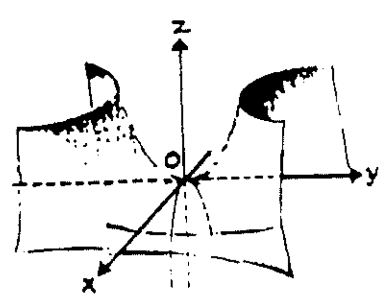


Fig. 10. A hyperbolic paraboloid like that at St. Mary's Cathedral. (*The Joy of Mathematics*, 171)

St. Mary's Cathedral in San Francisco, built and designed by Paul Ryan, John Lee, Pier Luigi Nervi, and Pietro Bellaschi, is a 2135 cubic foot hyperbolic paraboloid cupola with walls 200 feet off the ground and weighing about 36 million pounds. The equation of such a hyperbolic paraboloid is y^2/b^2-x^2/a^2=z/c^2 where a and c are greater than 0, and c is not equal to 0.[27]

The ceiling of the Capitol is a parabolic ceiling (designed in 1792 by Thornton). John Quincy Adams, while still in the House of Representatives, used the following property of a parabola to listen in on other Representatives. Sound bounces off the reflector of the dome, parallel to the opposite reflector and then bounces to its focal point. In this way Adams was able to listen to people across the room who were standing at the focus. Noises coming from elsewhere will not affect the clarity of reflected sound in a parabolic room. Tour guides at the Capitol Building demonstrate this property for visitors.[28]

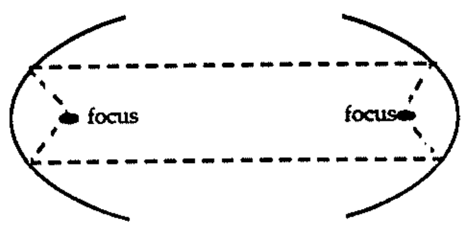


Fig. 11. Parabolic reflectors at the Capitol. (*The Joy of Mathematics*, 23)

Going ballistic:

Galileo found that the path of a cannonball rolled off the end of a plank was the descending branch of a parabola. The trajectory of a projectile fired in a vacuum from an inclined gun barrel would include both the ascending and the descending branches of a parabola.[29]

Beyond a shadow of doubt:

No matter where the light source, the shadow of an ellipsoid is always an ellipse. (This is because the intersection of any ellipsoid and a plane is always an ellipse.)[30]

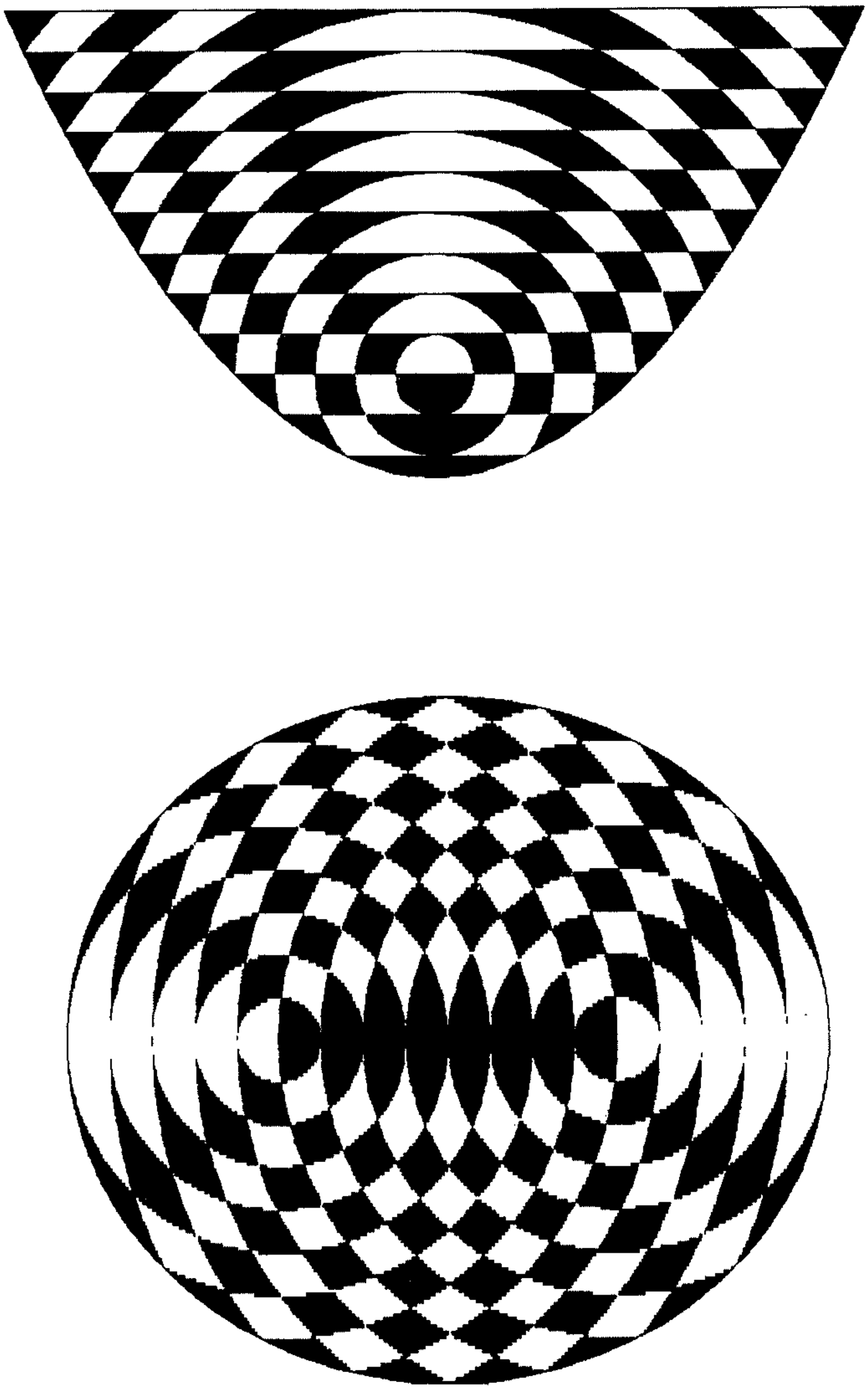
That's a moiré:

On the next page are two moiré patterns -- the intersections of two families of circles where each family shares a center and the radii are integers. The intersections lie on hyperbolas and ellipses. When the distance between the families' centers is infinity you have a family of circles and a family of parallel lines - This makes parabolas. When the distance is zero you get circles.[31]

What does it mean:

Not only can you define conics in terms of slices of a cone, quadratic equations, by geometric definitions, and using eccentricity, but you can also define them by using geometric mean. (If $z^2 = xy$ then z is the geometric mean between x,y.) If x,y,z are homogeneous coordinates in the plane than that equation can be used to define any non-degenerate conic.[32]

Fig. 12. Moiré patterns form an ellipse and a hyperbola.



Completely hyper:

Here's a picture of a complete monster.[33]

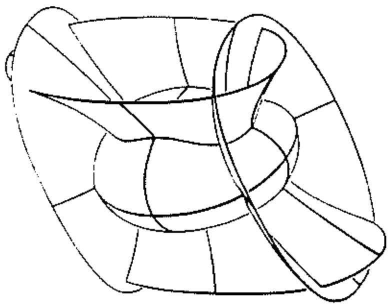


Fig. 13. The monster, an orthogonal surface of confocal conics. (Say that three times quickly.) (*The Penguin Dictionary of Curious and Interesting Geometry*,. 166)

And in case anyone ever asks, quadratic surfaces in n-dimensional space are described by x.(A.x) + b.x + c = 0 where x = position vector, A = symmetric tensor (basically a symmetric matrix), b = constant vector, and c = constant scalar. A.x is the vector obtained by multiplying tensor (matrix) A by vector x and u.v is the scalar product of vectors u and v.[34]

Oy gevalt:

(Warning -- reading this paragraph may prove harmful to your health.) We can also relate the shape of the conic to the eigenvalues and eigenvectors of the matrix A. (In case you're not all that familiar with eigenvalues and vectors, here's a brief definition: eigenvalues λ so $AX=\lambda X$, scalar λ where there are nonzero solutions if A is an N*N matrix and there is some vector X. λX is the eigenvector.[35])

1) $A = 0$ --> Line (in most cases)

2) A = nonzero scalar*(identity matrix) --> Circle (if a curve exists)

3) Eigenvalues of A are unequal but have the same sign --> Ellipse (again if the curve exists). If the eigenvalues are equal, then for a symmetric matrix we must have a scalar times the identity matrix, and the case reverts to 1) or 2).

4) One eigenvalue is zero, the other nonzero -->Parabola, 2 parallel lines, one line, or no curve. The parabola is the usual case; the others occur only for special cases. Note that the parallel lines are a cylindrical section; a cylinder is one limiting case of the cone.

5) Eigenvalues have opposite signs --> Hyperbola or 2 intersecting lines.[36]

Equations and graphs:

In addition to being graphed by quadratic equations, conics can be expressed as $(p^2 + q^2)[(x-a)^2 + (y-b)^2] = e^2(px + qy + r)^2$ where e is the eccentricity, (a, b) the focus, and $px + qy + r$ is the equation of the directrix of the conic. In vertex form -- $y^2 = 2px - (1- e^2)x^2$.[37]

The parametric equation of an ellipse is $x = a \cos \theta$ $y = b \sin \theta$. The foci are located at $(\pm ae, 0)$ and ae = square root ($a^2 - b^2$).[38] The parametric equation of a hyperbola is $x = a \sec \theta$, $y = b \tan \theta$.[39] The parametric equation of a parabola is $x = at^2$, $y = 2$.[40]

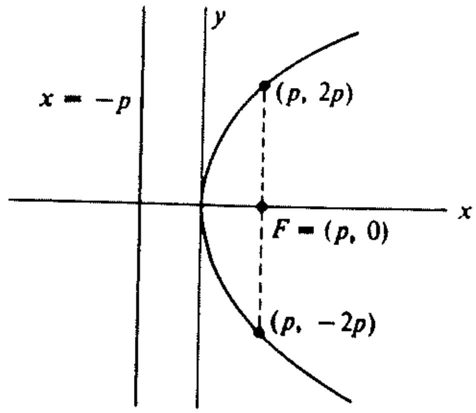


Fig. 14. (*Foundations of Modern Math*, 124)

The equation of an ellipsoid (in plain old everyday space) is x^2 /a^2 + y^2/b^2 + z^2/c^2 = 1.[41] The equation of a hyperboloid of one sheet is x^2 /a^2 + y^2/b^2 - z^2/c^2 = 1, and of two sheets x^2 /a^2 - y^2/b^2 - z^2/c^2 = 1.[42] Elliptical and hyperbolic paraboloids are found by the equation x^2/a^2 ± y^2/b^2 = 2cz.[43]

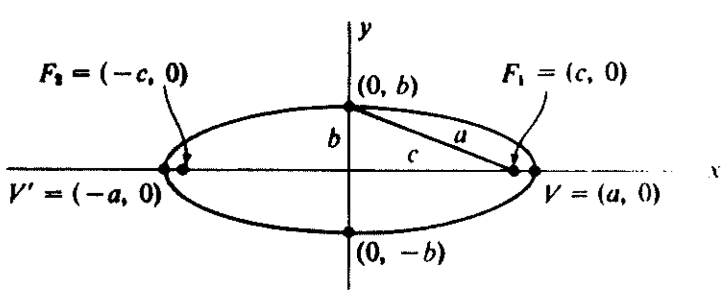


Fig. 15. (*Foundations of Modern Math*, 127)

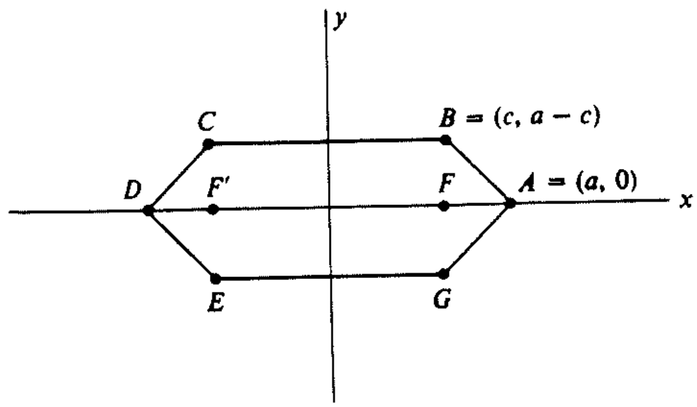


Fig. 16. A non-euclidian ellipse.
(*Foundations of Modern Math,* 138)

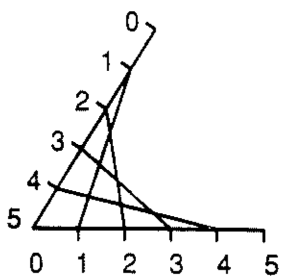


Fig. 17. A parabola.
Penguin Dictionary, 171)

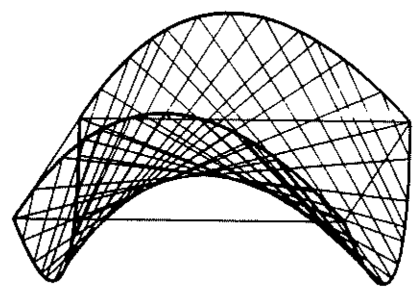


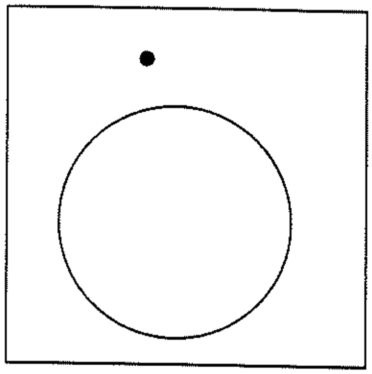
Fig. 18. A hyperbolic paraboloid.
(*Penguin Dictionary,* 111)

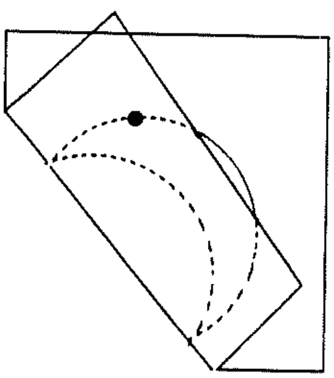
There are lots of different methods for graphing conic sections. You can construct them, use equations (like those above), use geometric definitions, use architectural and geometric tools.... the list is almost endless. Here are a few methods that most people aren't familiar with: Ellipses can be formed using their directrices, a rectangle's midpoints, or a squashed circle.[44] Paper folding in a circle can create ellipse or hyperbolas.[45] Another method of graphing ellipses was found by Leonardo da Vinci.[46] An instrument called the trammel can also be used to form ellipses.[47]

PAPER FOLDING

Here's how to fold a hyperbola.

- Begin with a circle drawn on a sheet of paper.
- Select a point in the circle's exterior and mark its location with a dot.
- Fold a point of the circle's boundary to the dot as shown and crease the paper.





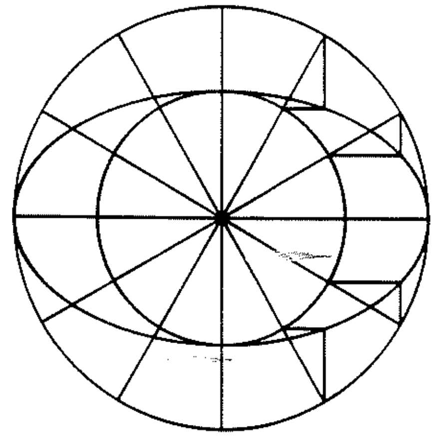
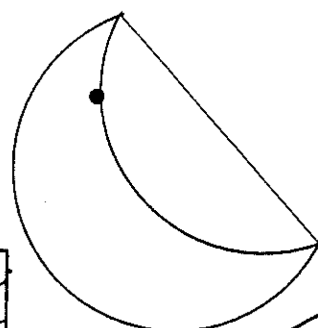
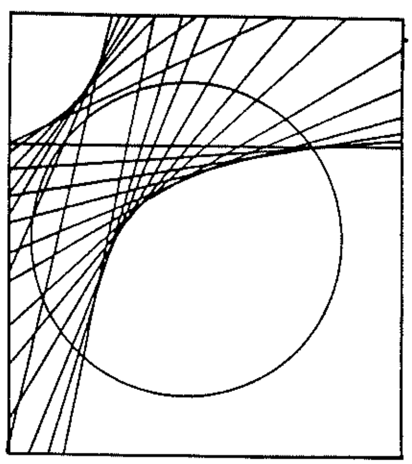


Fig. 19. An ellipse is a squashed circle. (*Penguin Dictionary*, 65)



- Continue this process of folding and creasing around the entire boundary of the circle.



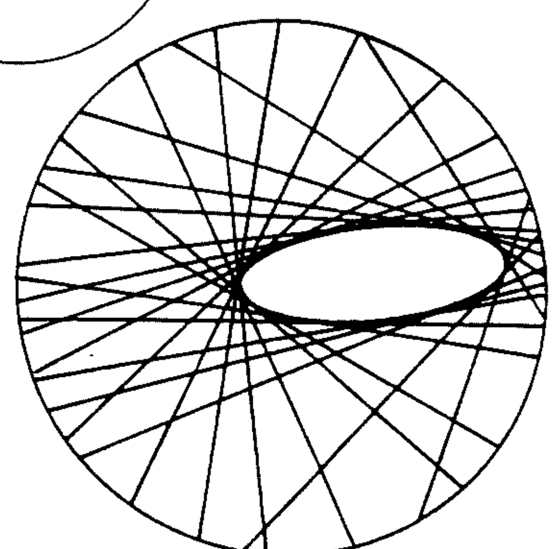


Fig. 20. and Fig. 21. A hyperbola and an ellipse (*More Joy of Mathematics*, 22-23)

- Start with a paper circle.
- Select a point in the interior of the circle that is not the center. Mark its location with a dot.
- Fold and crease the circle so a point of its boundary lands on the dot.
- Continue the above process working your way around the circle's boundary.

Eventually the shape of the ellipse will form.

To draw an ellipse in a rectangle, divide one half of each of the sides and one half of the line joining the mid-point of a pair of opposite sides into an even number of parts, and find the intersections of the lines joining X and Y to the marked points.

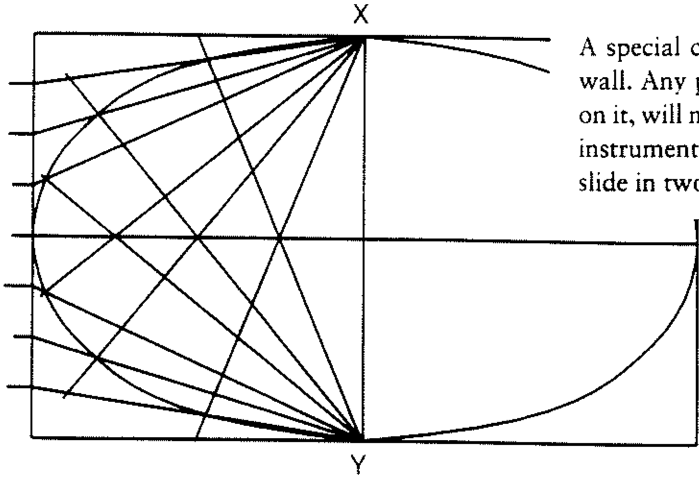


Fig. 22. **An ellipse in a rectanagle.** (*Penguin Dictionary*, 64)

A special case of this construction occurs when a ladder slips against a wall. Any point on the ladder, such as the foot of a person still standing on it, will move in a portion of an ellipse. This is the basis of a commercial instrument for drawing an ellipse using trammels. Two points of a rod slide in two grooves, and the path of a point on the rod is an ellipse.

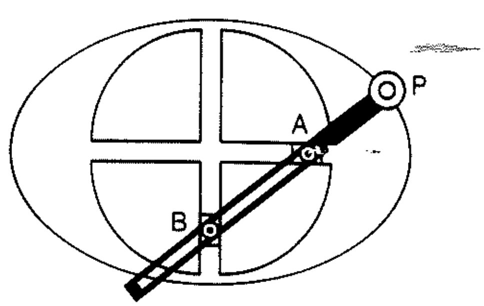


Fig. 24. Drawing an ellipse with a trammel. (*Penguin Dictionary*, 66)

The following method of drawing an ellipse was discovered by Leonardo da Vinci. Cut out a triangle ABC. Draw two axes, which need not be perpendicular, on a piece of paper, and move the triangle so that one vertex moves along one line and another moves along the second line. The path of the third vertex will be an ellipse.

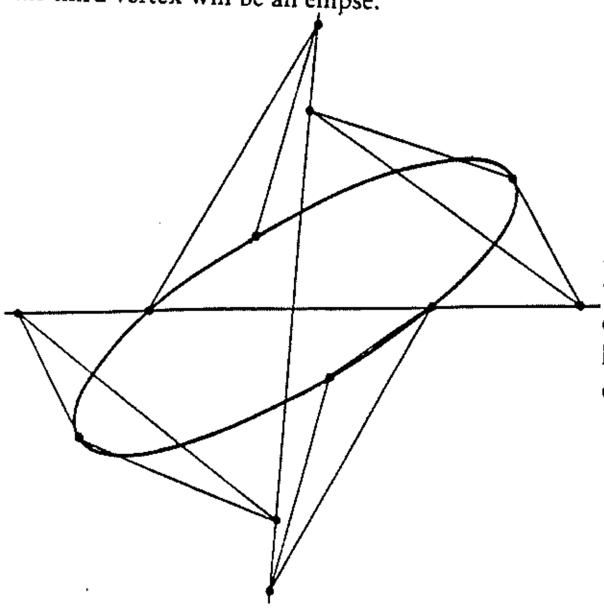


Fig. 23. Leonardo daVinci's method. (*Penguin Dictionary*, 65 and *More Joy of Mathematics*, 265)

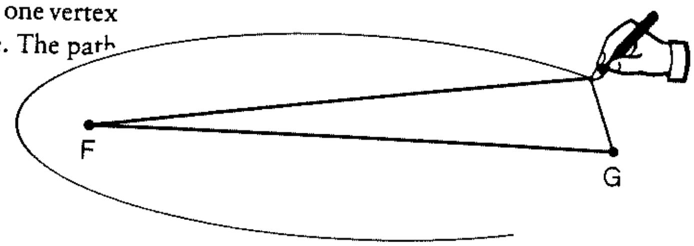


Fig. 25. The geometric definition of an ellipse is demonstrated using string and a pencil. (*Penguin Dictionary*, 64)

The hyperbola can be drawn mechanically by a method similar to, but less simple than, that for the ellipse. Let AX be a rod rotating about A, which will be one focus of the hyperbola. Attach a length of string to the end of the rod and to the other focus, B, and keep it taut by a pencil, shown here at P, held against the rod. As the rod rotates, P traces out one branch of a hyperbola.

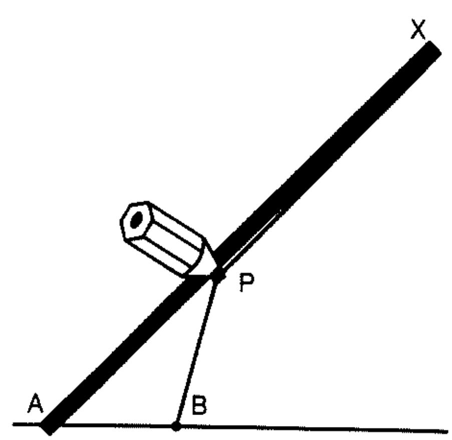


Fig. 26. A mechanically drawn hyperbola. (*Penguin Dictionary*, 107)

Another (more complicated) instrument can be used to graph a hyperbola. A rod with a string attached rotates about one of the foci. When the rod rotates, the arc that it creates is a hyperbola.[48] (See Figure 26)

Hyperbolic graphs can be used to look at conics. Define the distance to be equal to $|x_2 - x_1| + |y_2 - y_1|$. The graph of a circle looks like a square, that of an ellipse looks like a hexagon (Figure 16), and the graph of a parabola looks like a pentagon that is missing a side.[49]

Vocabulary words of the day:

Lattice Rectum. The lattice rectum of a conic is the chord that passes through the focus and is perpendicular to the major axis of a conic . It has a length of 2p.[50]

Cassinian Oval. If a point moves so that the product of its distances from the foci is constant, its path is a Cassinian oval, cross-sections of elliptical torus.[51]

Confocal Conics. Given any pair of points, there are an infinite number of ellipses and hyperbolas with these points as foci. No ellipse meets another, nor does a hyperbola, but every ellipse meets every hyperbola at right angles. Given a point and a line, there are an infinite number of parabolas.[52]

Orthogonal Surfaces. Two or three families of 2- or 3-dimensional curves that intersect every member of the other family(ies) pependicularly. The hyper-monster shown in Figure 13 is analogous to the family of confocal conics with one family being an ellipsoid family, the second being hyperboloids of one sheet, and the third being hyperboloids of two sheets.[53]

Pencil of conics. There is a unique conic through five point, or touching five lines, and an infinite family of conics touching four lines.[54]

Steiner's Roman Surfaces. y^2z^2 + z^2x^2 + x^2y^2 + xyz = 0, pinch-points, four planes, heptahedron form.[55] Another way to look at surfaces deals with Boy surfaces. Romboy (a portmanteau of Roman and Boy) homotopy is based on the idea of rotating ellipses through space.[56]

Fig. 27. A Romboy homotopy. (*Islands of Truth,* 41)

And here's a short story to end this paper:

> Once upon a time, there were 17 yellow pigs who lived on the surface of a hyperdonut. And near them in the terrible land of eigenvectors and quaternions lived a big, bad pink elephant and his 22 children. One day the big, bad elephant said he would sneeze, and sneeze, and blow their house down, but he couldn't so he tried to come down their Klein bottle chimney, but that's another story. This story involves just one of the pigs, Sgip Wolley, and one of the elephants, Tnahpele Knip, who were sort of friends (or as close to friends as a pig and an elephant could be). One day they were walking down the hyperbolic paraboloid hills, where Dandelins (not to be confused with Dandelions) grew wild that separated their villages. Tnahpele was telling a story about a trunk (tusk, tusk) and Sgip was squealing with delight and snorting with laughter. Just then, however, Tnahpele's father came along and told them that they could not be with each other because yellow pigs and pink elephants did not get along. So after that Sgip and Tnahpele never saw each other again. But every 17*23 days, if you look at a Dandelin sphere, you'll see a pink trunk and a little yellow curly tail sticking out, and you'll know that the friendship of the two young animals never really ended.[17*23]

That's all for the interesting and semi-interesting facts (well, there are a lot more, but I couldn't get the books that had them in time). Here's an interesting thought about conicoids and hyper-Dandelin spheres .... or not. Now I have thoroughly exhausted the topic of Dandelin spheres (and anything else related to them in any way, shape, or form).

Well, that gets as close to concluding this topic as I think that I will get. Hope you have enjoyed these excursions in conic sections. Now suppose you graph all of the conics on a torus to form envelopes, or better yet on a Mobius strip or Klein bottle.......

Just when I thought that I was really done!
A page of Afternotes

Today I received Daniel Scher's book, Exploring Conic Sections with *The Geometer's Sketchpad* (Berkeley: Key Curriculum Press, 1995). The following information comes from his book.

Ellipses:

The two-pins-and-a-string construction, the folded-circle construction, and the trammel are already addressed in my paper, but here's a little more on the trammel. Here's how to make a simple trammel with a ruler, tape, a pencil, and a protractor -- use the tape to mark three spots on the ruler. (I suggest spots at 17 cm, 23+23/17 centimeters, and 17+34/23 centimeters, but those are my entirely random numbers.) Get a sheet of paper and use the protractor to draw two perpendicular lines (the axes of your plane). Position your ruler so that your middle point is on the origin and the ruler lines up with the x-axis. Slide the ruler so that your midpoint is on the y-axis and one of the other points is still on the x-axis. Continue doing this marking all points that the third point on your ruler is passing. These points should trace out one quadrant of an ellipse.

Another way to construct an ellipse is courtesy of Frans von Schooten. Here's how with a straw, a pencil, tape, and a sheet of paper -- bend the straw in half and mark a point on one side with the tape. Draw a horizontal line on the paper and lay the straw so that it lies flat along the line and mark the position of your point on the paper. Hold the side without the mark in place while pushing the other side a little inward along the line toward the other side of the straw. Continue this while tracing out the point to get a quarter of an ellipse.

Parabolas:

I realize that while I showed paper folding methods for the ellipse and hyperbola, I did not include those methods for a parabola or circle. To fold a parabola, mark a point near the bottom-middle of a sheet of paper. Fold the paper and crease lines that contain this point. Each of these lines is tangent to a parabola. Another way to make a circle or any other conic section is to use a straightedge in a way similar to creases. To make a circle, pick a point and place the ruler so that one edge is tangent to that point. Draw a line using the other side of the ruler. Continue this process to get a circle with a radius the length of the width of the ruler. To prove that paper folding about a point does generate a parabola, you need to use geometric means by finding chords on a circle that are perpendicular to a segment at a certain point.

Hyperbolas:

While an ellipse has a constant sum of distances from two points and can be shown as the intersection of circles with two different radii, a hyperbola can be similarly shown by making a small change in the intersecting-circles construction. A hyperbola can also be folded where the center of the circle is one focus and the point outside of the circle that you pick is the other focus.

Pascal's Theorem:

Pick any six points (a, b, c, d, e, and f) on a conic. Draw segments ab, bc, cd, de, ef, and fa. Label the intersections of ab and de as point A, bc and ef as point B, and cd and fa as point C. The points A, B, and C are collinear. Using the converse of this theorem, a conic can be constructed through any five points as long as no three are collinear.

Endnotes

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[10] David Wells, The Penguin Dictionary of Curious and Interesting Geometry, (London: Penguin Books, 1991) 48.

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[12] HopD, letter to the author, 7 March 1995.

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[16] Ivars Peterson, The Mathematical Tourist: Snapshots of Modern Mathematics (New York: W.H. Freeman and Company, 1988) 165.

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[22]Peterson, Mathematical Tourist 12.

[23]Pappas, Magic of Mathematics 184.

[24] Matt Pgh, letter to the author, 10 March 1995.

[25]Pappas, Magic of Mathematics 18-19.

[26] Theoni Pappas, The Joy of Mathematics: Discovering Mathematics All Around You, (San Carlos, California: Wide World Publishing/Tetra, 1989) 196-197.

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[32]John Harper, letter to the author, 5 March 1995.

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[38] Borowski and Borwin, 188.

[39]Borowski and Borwin, 272.

[40]Borowski and Borwin, 432-433.

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[46]Pappas, More Joy of Mathematics 265.

[47]Wells, 64-66.

[48]Wells, 107.

[49]Eyes, 136-139.

[50] Borowski and Borwin, 331.

[51]Wells, 26.

[52]Wells, 39.

[53]Wells, 166.

[54] Wells, 175.

[55] Wells, 247.

[56]Ivars Peterson, Islands of Truth: A Mathematical Mystery Cruise. (New York: W.H. Freeman and Company, 1990) 40-41.

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